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Toric Geometry and Enhanced Gauge Symmetry of F-Theory/Heterotic Vacua

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ABSTRACT

We study F-theory compactified on elliptic Calabi–Yau threefolds that are realised as hypersurfaces in toric varieties. The enhanced gauge group as well as the number of massless tensor multiplets has a very simple description in terms of toric geometry. We find a large number of examples where the gauge group is not a subgroup of $E_8 \times E_8$, but rather, is much bigger (with rank as high as 296). The largest of these groups is the group recently found by Aspinwall and Gross. Our algorithm can also be applied to elliptic fourfolds, for which the groups can become very large indeed (with rank as high as 121328). We present the gauge content for two of the fourfolds recently studied by Klemm *et al.*

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Contents

1. **Introduction**

The dualities of String Theory have been the subject of extensive study during the last two years. Of particular interest to us here is the duality [1,2]

$$\mathrm{Het}[K3 \times T^2, G] = \mathrm{IIA}[\mathcal{M}] \tag{1.1}$$

between a $(0,4)$ heterotic compactification on $K3 \times T^2$ with gauge group G, and a type IIA compactification on a Calabi–Yau manifold, $\mathcal{M}$. This is a very exciting arena in which to explore nonperturbative phenomena in String Theory and has been the focus of much recent work [3-7].

A large class of Calabi–Yau manifolds can be realised as hypersurfaces in toric varieties and, in virtue of a construction of Batyrev [8,9], these have a nice description in terms of a dual pair (Δ, ∇) of reflexive polyhedra. This being so we may regard the Calabi–Yau manifold $\mathcal{M}$ as being specified by a polyhedron $\mathcal{M} = \mathcal{M}_\Delta$. Thus, it is natural to suppose that the polyhedron determines the gauge group G that appears on the heterotic side. This was the point of view adopted in [10-12]. In these articles a first dictionary between perturbative symmetry restoration on the heterotic side and toric data was established. The duality (1.1) has far-reaching consequences. It is believed to apply most directly to Calabi–Yau manifolds that are both elliptic and $K3$ fibrations. Many of these can be described by reflexive polyhedra and so it is natural to suppose that there is a correspondence $G = G_\nabla$. This was shown to be the case for certain simple cases in [10-12], the correspondence being stated most simply in terms of the dual polyhedron ∇.

The point we make here is that if we take (1.1) seriously, then there should be a Heterotic theory for a great many polyhedra ∇ that correspond to elliptic and $K3$ fibrations. To appreciate the consequence of this, consider the Calabi–Yau manifold

$$\mathcal{M}_{E_8} \;=\; \mathbb{P}_4^{(1,1,12,28,42)}[84]$$

which is an elliptic $K3$ fibration and corresponds to a gauge group E_8. Now $\mathcal{M}_{E_8}$ has Hodge numbers $h_{11} = 11$ and $h_{21} = 491$. The Hodge number h_{11} is related to the rank of the group and the number of tensor multiplets, n_T, by a general relation

$$h_{11} = \mathrm{rank}(G) + n_T + 2 \tag{1.2}$$

which is satisfied in this case with $G = E_8$ and $n_T = 1$. Now it turns out, by virtue of the work of Avram *et al.* [13] that the mirror of $\mathcal{M}_{E_8}$ is also an elliptic-$K3$ fibration. The mirror, however, has $h_{11} = 491$ which suggests, in virtue of (1.2) that G will have large rank. In fact this is so. This is the manifold studied by Aspinwall and Gross [14] who find that G is a group of rank 296

$$G = E_8^{17}{\times}F_4^{16}{\times}G_2^{32}{\times}SU(2)^{32} \quad \text{and} \quad n_T = 193.$$

From the toric perspective the group is large because the dual polyhedron which was 'small' for the cases considered in [10-12] has now many points. The purpose of this paper is to present an algorithm that allows the group to be read off from the dual polyhedron. Typically one obtains in this way groups of large rank corresponding to the fact that (1.1) obtains for many manifolds $\mathcal{M}$, and the typical $\mathcal{M}$ has large h_{11}. The mirrors of the manifolds discussed in [10] are a case in point and provide many examples for which large gauge groups arise.

The organisation of this paper is as follows. In §2 we review relevant aspects of toric geometry, principally the construction of Refs. [15-17] of F-Theory duals of Heterotic vacua in six dimensions and the observation of [13] that the mirror of an elliptic Calabi–Yau manifold is frequently also an elliptic Calabi–Yau manifold. In §3, we describe the algorithm which allows us to read off the vector and tensor multiplet content of the effective theory from the toric data. In §4, we illustrate this approach with some examples. The procedure can also be applied to elliptic fourfolds, and we present the gauge content for two of the fourfolds studied by Klemm *et al.* [18]. §5 discusses subtleties which arise in the toric picture. §6 summarizes our results. Tables giving the groups and tensor multiplet content of the models that we study are given in the Appendix.

2. F-Theory Compactified on Elliptic Calabi–Yau Threefolds

In this section, we review $N = 1$ vacua in six dimensions that result from compactification of F-theory on elliptic Calabi–Yau threefolds, largely following [15,16].

Recall that an elliptic Calabi–Yau threefold can be described by the Weierstrass equation

$$y^2 = x^3 + f(z, z')x + g(z, z'), \tag{2.1}$$

where z and z' are affine coordinates on the base. At the divisors on the base given by the zero loci of the discriminant,

$$D = 4f^3 + 27g^2, \tag{2.2}$$

the torus degenerates. In many cases when this happens the effective four or six dimensional effective theory develops a nonabelian gauge symmetry. The singularities determine the gauge group and matter content of the F-theory compactification. The type of the singularity, and hence the resulting gauge group, depends on the form of the polynomials f and g. A dictionary relating the singularities of such elliptic fibrations and gauge symmetry enhancement was given in [17].

One can arrive at a singular locus by adjusting the coefficients in the polynomials f and g, or, in other words, by varying the complex structure parameters of the threefold. It is then possible to resolve the singularities by sequences of blow-ups, *i.e.,* by varying the Kähler class parameters. The smooth Calabi–Yau manifold that results still contains all the information about the enhanced gauge symmetry. Moreover, in many relevant cases, it can be conveniently represented in terms of the toric data as was shown in [10-12].

Suppose we are interested in F-theory compactifications dual to $E_8 \times E_8$ heterotic models on a $K3$ with instanton numbers $(12+n, 12-n)$ in the two E_8's and blowups thereof. Then the starting point is the hypersurface in the toric variety defined by the data displayed in Table 2.1 [15]. Namely, start with homogeneous coordinates s, t, u, v, x, y, w, remove the loci $\{s = t = 0\}$, $\{u = v = 0\}$, $\{x = y = w = 0\}$, take the quotient by three scalings (λ, μ, ν) with the exponents shown in Table 2.1 and restrict to the solution set of (homogeneous version of) Eq. (2.1).

	s	t	u	v	x	y	w	degrees
λ	1	1	n	0	$2n{+}4$	$3n{+}6$	0	$6n+12$
μ	0	0	1	1	4	6	0	12
ν	0	0	0	0	2	3	1	6

Table 2.1: The scaling weights of the elliptic fibration over $\mathbb{F}_n$.

These data define a Calabi–Yau threefold. F-theory compactified on this manifold is dual to the $E_8 \times E_8$ heterotic model with instanton numbers $(12+n, 12-n)$ and maximally Higgsed gauge group. We can now construct the Newton polyhedron describing this hypersurface. Concretely, first we find all possible nonnegative powers of our homogeneous variables compatible with the constraints. We get in this way a number of points in $\mathbb{R}^7$. Since there is a $(\mathbb{C}^*)^3$ action these points actually lie in a four-dimensional subspace. Having chosen a basis we take the convex hull of the points and obtain our Newton polyhedron Δ.

We can vary the complex structure parameters to introduce singularities into our Calabi–Yau threefold and nonabelian enhanced gauge symmetry into the corresponding effective theory. For example, it was shown in [16] that if we introduce a curve of singularities at $z = 0$[1], then, on the heterotic side, this will have the effect of unhiggsing a certain subgroup of the first E_8. Suppose now that we want to construct the corresponding polyhedron. To do it in the most direct way, it is easiest to go back to our seven-dimensional points which simply give us various powers of the homogeneous coordinates and use the results of [17] which relate the types of singularities to the degrees of vanishing of certain polynomials[2] on the base of the elliptic fibration. Thus, in toric language, the introduction of a curve of singularities at $z = 0$ means simply eliminating a certain number of

[1] The base of the elliptic fibration $\mathbb{F}_n$ is a fiber bundle over $\mathbb{P}_1$ with fiber $\mathbb{P}_1$. We denote, following [16], the affine coordinate on the base by z', and the affine coordinate on the fiber by z.

[2] The authors of [17] rewrite the Weierstrass equation in a more general form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

and the polynomials are just the $a_i = a_i(z, z')$.

seven-dimensional points, which in turn results in the disappearance of the corresponding four-dimensional points from the Newton polyhedron Δ. Since the Newton polyhedron is diminished, the dual polyhedron ∇ acquires some additional points in the process. It is useful to emphasize here that the manifolds corresponding to the polyhedra which result are *smooth* and correspond, upon compactification to four dimensions on a T^2, to a theory in the Coulomb phase.

It was observed in [10] that it is the dual polyhedron which exhibits a regular structure which makes possible, in particular, to determine the enhanced gauge symmetry given ∇. It was noticed there that in all the examples of heterotic/type II dual pairs the $K3$ and elliptic fibration structure shows itself in the existence of three- and two-dimensional reflexive subpolyhedra, respectively, inside the dual polyhedron of the Calabi–Yau manifold. Moreover, the three-dimensional reflexive subpolyhedron which was conjectured to represent the generic $K3$ fiber was shown to contain the information about the part of the total gauge group (the only part in the examples considered in [10]) which has perturbative interpretation on the heterotic side.

3. **Identifying the Groups**

3.1. Generalities

In this section, we describe the algorithm that allows us to read off the gauge content from the toric data. First we introduce a generalization of the conjecture of Ref. [10], using the results of [13] to state this for Calabi–Yau manifolds that are described by reflexive polyhedra, the integral points of the polyhedra being points in a lattice Λ. It has been shown there that in order for a Calabi–Yau n-fold to be a fibration with generic fiber a Calabi–Yau $(n-k)$-fold it is necessary and sufficient that[3]

> (i) There is a projection operator Π: $\Lambda \to \Lambda_{n-k}$, where Λ_{n-k} is an $n-k$ dimensional sublattice, such that $\Pi(\Delta)$ is a reflexive polyhedron in Λ_{n-k}, or
>
> (ii) There is a lattice plane in $V_{\mathbb{R}}$ through the origin whose intersection with ∇ is an $n-k$ dimensional reflexive polyhedron, *i.e.* it is a slice of the polyhedron.

(i) and (ii) are equivalent conditions. In case (i) the polyhedron of the fiber appears as a *projection* while in case (ii) it appears as an *injection*, the projection and the injection being related by mirror symmetry (see Figure 3.1). In particular, if the polyhedron of the $(n-k)$-dimensional Calabi–Yau manifold exists as both a projection and an injection (*e.g.,* the image of the projection coincides with a slice of the Δ by a plane as sketched in the second row of Figure 3.1), then the intersection in ∇ is also a certain projection implying that the mirror manifold is a fibration with an $n-k$ dimensional Calabi–Yau manifold as the typical fiber. If (i) or (ii) hold there is also a way to to see the base of the fibration torically[19]. The hyperplane H generates a $n-k$ dimensional sublattice of V. Denote this lattice V_{fiber}. Then the quotient lattice $V_{\text{base}} = V/V_{\text{fiber}}$ is the lattice in which the fan of the base lives. The fan itself can be constructed as follows. Let Π_B be a projection operator acting in V, of rank $\dim(V)-2$, such that it projects H onto a point. Then $\Pi_B(V) = V_{\text{base}}$. When Π_B acts on ∇ the result is a k dimensional set of points in V_{base} which gives us the fan of the base if we draw rays through each point in the set.

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[3] We denote, as is standard, the lattice dual to Λ (where Δ lives) by V, and its real extension by $V_{\mathbb{R}}$.

Recall now that the Hodge numbers for three-dimensional Calabi–Yau hypersurfaces are given by

$$
\begin{aligned}
h_{21} &= \mathrm{pts}(\Delta) \ - \sum_{\mathrm{codim}(\theta)=1} \mathrm{int}(\theta) \ + \sum_{\mathrm{codim}(\theta)=2} \mathrm{int}(\theta)\mathrm{int}(\tilde{\theta}) \ - \ 5, \\
h_{11} &= \mathrm{pts}(\nabla) \ - \sum_{\mathrm{codim}(\tilde{\theta})=1} \mathrm{int}(\tilde{\theta}) \ + \sum_{\mathrm{codim}(\tilde{\theta})=2} \mathrm{int}(\tilde{\theta})\mathrm{int}(\theta) \ - \ 5
\end{aligned}
\tag{3.1}
$$

where $\mathrm{pts}(\Delta)$ denotes the number of integral points of Δ, $\mathrm{int}(\theta)$ stands for the number of integral points interior to a face θ and similar quantities $\mathrm{pts}(\nabla)$ and $\mathrm{int}(\tilde{\theta})$ are defined for ∇. Equation (3.1) expresses the number of deformations of complex structure and Kähler classes in terms of the number of points of the polyhedra. The terms in these expressions that involve codimension-1 faces account, in the case of h_{21}, for the freedom to make redefinitions of the homogeneous variables, and in the case of h_{11}, for the singularities of the toric variety which do not intersect the hypersurface. We will call these points 'irrelevant'. The third terms in both equations are 'correction' terms, the numbers of deformations of the corresponding hypersurface which are not visible torically. (Note that in many cases it turns out to be possible to add a certain number of points to the polyhedron under consideration so that the correction vanishes.)

It is worth emphasizing that the statement about the possibility of seeing the base of the Calabi–Yau fibration torically holds provided the correction term in (3.1) vanishes. Otherwise we may miss some of the one-dimensional cones in the fan describing the base as a toric variety.

Suppose now that we are given an elliptic Calabi–Yau threefold. The theorem of [13] tells us that in this case it is possible to find a two-dimensional hyperplane H in $V_{\mathbb{R}}$ through the origin such that its intersection with ∇ is a two-dimensional reflexive polyhedron representing the typical fiber. Let us denote it by $\nabla^{\mathcal{E}} = \nabla \cap H$. Projecting ∇ with Π_B such that $\Pi_B(\nabla^{\mathcal{E}}) = (0,0)$ yields a set of points living in a two-dimensional lattice which is what we call V_{base}. We will denote this set of points Σ_B. Drawing a ray through from the origin $(0,0)$ through every other point gives us the fan of the base. Note that a ray may pass through more than one point and hence the number of rays, or one-dimensional cones, is generically less than the number of non-zero points in V_{base}. In many examples of [10-12], Σ_B coincides with a two-dimensional face of the dual polyhedron ∇ orthogonal to the hyperplane H.

The methods of toric geometry allow us to read off some topological invariants of B. In particular it is known [20] that for a nonsingular n-dimensional toric variety the Betti

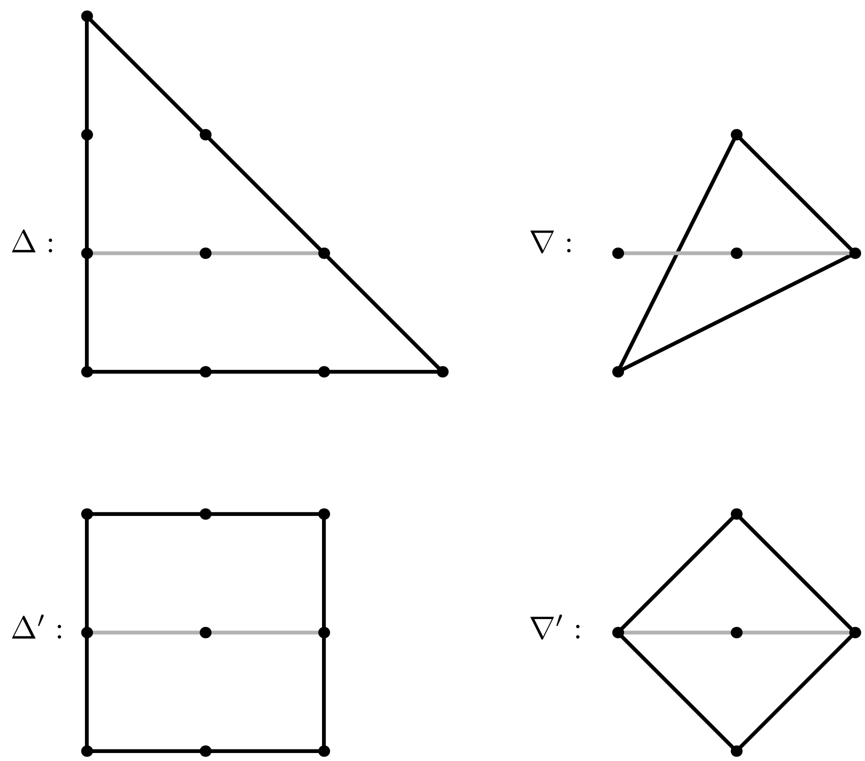

Figure 3.1: Two examples of fibrations visible as projections or slices of the polyhedra. Δ and ∇ are a dual pair of reflexive polyhedra corresponding to 1-dimensional Calabi–Yau manifolds. Δ encodes a zero-dimensional Calabi–Yau manifold ($= \mathbb{Z}_2$) as a slice (but not a projection). The dual polyhedron ∇ encodes the $\mathbb{Z}_2$ as a projection (but not an injection). According to the criterion of Ref. [13], ∇ is the Newton polyhedron of a $\mathbb{Z}_2$ fibration. Δ' encodes the $\mathbb{Z}_2$ fiber as both an injection as well as a projection, hence the mirror ∇' does too, so that both the manifold and its mirror are $\mathbb{Z}_2$ fibrations.

numbers are given by

$$b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} d_{n-i}, \tag{3.2}$$

where d_k is the number of k-dimensional cones in the fan. In our case (3.2) yields

$b_2 = d_1 - 2d_0$ or, since $h_{20}(B) = 0$ for a toric variety,

$$h_{11}(B) = d_1 - 2. \tag{3.3}$$

Compactifying Type IIA strings on our Calabi–Yau manifold yields a four-dimensional $N = 2$ vacuum in its Coulomb phase. This statement is intrinsically six-dimensional, meaning that the compactification of F-theory on (a blown-down version of) the same Calabi–Yau manifold yields an $N = 1$ vacuum in 6D which contains essentially the same information. We want to determine the spectrum of this six-dimensional theory. The first observation is that the number of massless tensor multiplets is already determined by (3.3) since it is simply [16]

$$n_T = h_{11}(B) - 1. \tag{3.4}$$

Our next task is to find the massless vector multiplet content. Comparison of (1.2) with (3.1) shows us that essentially each 'relevant' point in ∇ corresponds either to a massless tensor multiplet or to a massless vector in the Cartan subalgebra of G provided the correction vanishes. In view of (3.3) it is reasonable to conjecture that the points corresponding to tensor multiplets are those in Σ_B, moreover, since each one-dimensional cone in that projection may contain more than one point, we will claim that the 'tensor multiplet' points are those closest to the origin of Σ_B (apart from three of them since $n_T = d_1 - 3$). Thus, in order to find the vector multiplet content, we should try to sort out the rest of the 'relevant' points in ∇. It has been conjectured and illustrated by many examples in [10] that in cases dual to $E_8 \times E_8$ heterotic on a $K3$ the points corresponding to vectors in the Cartan subalgebra of G had the following properties.

1. Under the projection $\Pi_B\colon \nabla \to \Sigma_B$ the points corresponding to the subgroup of the first E_8 project onto points in Σ_B of the form $(0, -b)$ (in a certain basis), $b > 0$ and the points corresponding to the unbroken subgroup of the second E_8 project onto points of which can be written as $(0, c)$, $c > 0$.

2. Under the projection $\Pi_{\mathcal{E}}\colon \nabla \to H$ these two sets of 'relevant' points projected onto certain points of $\nabla^{\mathcal{E}}$, and the information about the precise nature of the unbroken group (or, in geometric terms, about the type of singularity along the corresponding divisor of B) was contained in the numbers $n_i = |{\Pi_{\mathcal{E}}}^{-1}(pt_i)|$, where $pt_i \in \nabla^{\mathcal{E}}$ (see Fig. 3.2) and, as usual, $|A|$ denotes the number of elements in set A.

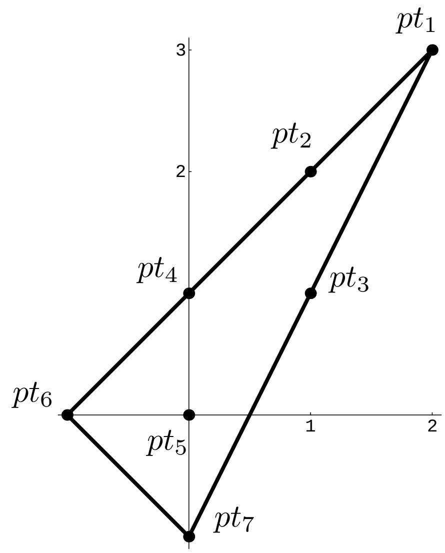

Figure 3.2: The polyhedron, ${}^2\nabla$, of $\mathbb{P}_2^{(1,2,3)}[6]$. The pt'_r are the points directly below the indicated points of the plot, with the number of primes denoting the depth.

3.2. The Algorithm

We are in a position now to propose our algorithm. Each one-dimensional cone in the fan of a toric variety corresponds to a divisor [20]. In our case the complex dimension of the base is two. So the divisors are complex curves. In the examples of [10] the one-dimensional cones described above (those containing points of the form $(0,-c)$ and $(0,b)$) corresponded to the divisors described by the equations $z=0$ and $z=\infty$, respectively. Let us call these cones (rays) R_1 and R_2. The points describing the unbroken subgroups of the first and second E_8 were those of the form $\Pi_B^{-1}(R_1)$ and $\Pi_B^{-1}(R_2)$. In geometrical language, they are the points describing the types of singularities along the two divisors, the corresponding set $\Pi_{\mathcal{E}}^{-1}$ encoding the type of the singularity. In general, the base may have many more curves of singularities than those two. Torically they will show themselves as one-dimensional cones. There is actually a singularity along a given divisor represented by the ray R_k provided the set $\Pi_B^{-1}(R_k)$ contains more than one point. To determine the actual type of singularity we have to act in exactly the same way as we would have acted

if the divisor in question had been given by $z = 0$. Namely we take $\Pi_B^{-1}(R_k)$ and project it onto $\nabla^{\mathcal{E}}$. The singularity type is then encoded in the corresponding numbers

$$n_{ik} = |\Pi_{\mathcal{E}}{}^{-1}(pt_i) \cap \Pi_B^{-1}(R_k)| \tag{3.5}$$

in exactly the same way as it would have been encoded had the singularity occurred along $\{z = 0\}$[4]. The dictionary between the numbers $\{n_i\}$ and singularity types was described in [10] (see Table 3.1) and carries over to this more general case.

3.3. Subtleties

There are subtleties in the picture described above which are worth mentioning. As was pointed out in [21] and [17], the gauge symmetry appearing in uncompactified dimensions turns out to be a subgroup of the singularity type ADE series group. These cases are due to the monodromy action on vanishing cycles if the monodromy happens to be an outer automorphism as opposed to a Weyl group element. These two cases were called 'non-split' and 'split', respectively, in [17]. Also different kinds of singularities may collide as discussed for example in [22,23], and when they do a new phase of the theory appears characterized by new tensor multiplets and enhanced gauge symmetry. We believe that these subtle points are already taken care of by the dual polyhedron and the points we observe provide us with information about the spectrum of the corresponding six-dimensional theory (and not quite, strictly speaking, about the types of singularities along divisors).

There are also certain additional subtleties that arise in the toric picture. This is because toric geometry does not always encode all the deformations as points in the polyhedra. There are often non-toric deformations coming from points interior to codimension-2 faces and points in the dual 1-faces (codimension-3 in the dual polyhedron). Thus some of the deformations are not explicitly seen as points in the polyhedron, but hidden away in the dual polyhedron. Thus our identification of the gauge theory is incomplete until we can interpret all these non-toric data. Furthermore, we also encounter situations where the polyhedron contains points which are interior to codimension-2 faces (and hence relevant), but which are interior to codimension-1 faces of the $K3$ fibers. For example, the point pt_5' in the E_8 top becomes relevant in some polyhedra that contain this top. This situation is tricky because it does not happen all the time — there are polyhedra containing the

[4] In all previous examples $pt_i \in \nabla^{\mathcal{E}}$. But in general there are cases in which we should regard some of pt_i's as belonging to H, not necessarily to $\nabla^{\mathcal{E}}$.

H	Bottom
$SU(1)$	$\{pt_1'\}$
$SU(2)$	$\{pt_1', pt_2'\}$
$SU(3)$	$\{pt_1', pt_2', pt_3'\}$
G_2	$\{pt_1'', pt_2', pt_3'\}$
$SO(5)$	$\{pt_1', pt_2', pt_4'\}$
$SU(4)$	$\{pt_1', pt_2', pt_3', pt_4'\}$
$SO(7)$	$\{pt_1'', pt_2', pt_3', pt_4'\}$
Sp_3	$\{pt_1', pt_2', pt_4', pt_6'\}$
$SU(5)$	$\{pt_1', pt_2', pt_3', pt_4', pt_5'\}$
$SO(9)$	$\{pt_1'', pt_2'', pt_3', pt_4'\}$
F_4	$\{pt_1''', pt_2'', pt_3', pt_4'\}$
$SU(6)$	$\{pt_1', pt_2', pt_3', pt_4', pt_5', pt_6'\}$
$SU(6)_b$	$\{pt_1', pt_2', pt_3', pt_4', pt_5', pt_7'\}$
$SO(10)$	$\{pt_1'', pt_2'', pt_3', pt_4', pt_5'\}$
$SO(11)$	$\{pt_1'', pt_2'', pt_3', pt_4'', pt_5'\}$
$SO(12)$	$\{pt_1'', pt_2'', pt_3', pt_4'', pt_5', pt_6'\}$
E_6	$\{pt_1''', pt_2'', pt_3'', pt_4', pt_5'\}$
E_7	$\{pt_1'''', pt_2''', pt_3'', pt_4'', pt_5'\}$

$SU(6)_c$	$\{pt_1', pt_2', pt_3', pt_4', pt_5', pt_6', pt_7'\}$
$SO(13)$	$\{pt_1'', pt_2'', pt_3', pt_4'', pt_5', pt_6''\}$
E_{6B}	$\{pt_1''', pt_2'', pt_3'', pt_4', pt_5', pt_7'\}$
E_{7B}	$\{pt_1'''', pt_2''', pt_3'', pt_4'', pt_5', pt_6'\}$
E_8	$\{pt_1^{(6)}, pt_2^{(4)}, pt_3''', pt_4'', pt_5'\}$

H	Bottom
$SU(2)_b$	$\{pt_2'\}$
$SU(2)_c$	$\{pt_4'\}$
$SU(2)_d$	$\{pt_6'\}$
$SU(2)\times SU(2)$	$\{pt_2', pt_4'\}$
$(SU(2)\times SU(2))_b$	$\{pt_4', pt_6'\}$
$SU(3)\times SU(2)$	$\{pt_2', pt_4', pt_5'\}$
$(SU(3)\times SU(2))_b$	$\{pt_3', pt_5'\}$
$(SU(3)\times SU(2))_c$	$\{pt_5'\}$
$SO(5)\times SU(2)$	$\{pt_2', pt_4', pt_6'\}$
$G_2\times SU(2)$	$\{pt_2', pt_4'', pt_5'\}$
$SU(4)\times SU(2)$	$\{pt_2', pt_4', pt_5', pt_6'\}$
$SO(7)\times SU(2)$	$\{pt_2', pt_4'', pt_5', pt_6'\}$
$SO(9)\times SU(2)$	$\{pt_2', pt_4'', pt_5', pt_6''\}$
$SU(3)_b$	$\{pt_3'\}$
$SU(3)_c$	$\{pt_7'\}$
$SU(3)\times SU(3)$	$\{pt_3', pt_5', pt_7'\}$

$G_2\times SU(3)$	$\{pt_3', pt_5', pt_7''\}$
$F_4\times SU(2)$	$\{pt_2', pt_4'', pt_5', pt_6'''\}$

Table 3.1: The table on the left gives the bottoms containing pt_1' formed by adding the points $pt_r^{(j)}$ to ${}^2\nabla$. In each case the points of ${}^2\nabla$ are understood and the points that are written are the lowest members of columns. Thus pt_2''' for example implies the presence of pt_2'' and pt_2'. The bottoms that appear in the lower block correspond to nonperturbatively realised groups. The table on the right gives the bottoms that do not contain pt_1'. The bottoms that are given in the lower block again correspond to groups that are realised nonperturbatively.

E_8 top where pt_5' is irrelevant, and there are others containing the E_8 top in which pt_5' is relevant. There are even polyhedra where both kinds of E_8 tops occur. Thus our algorithm is incomplete until we specify how to handle such cases. While we do not have a general theorem that achieves this, we have been able to study many such cases and have found consistent patterns that allow us to treat this and many similar situations which arise in these polyhedra. The prescription for handling such situations is best described by giving examples, which we postpone until §5.

4. **Examples**

We turn now to the application of our algorithm to some examples. Our first example is the well-known case of 'simple' (in the terminology of [22]) point-like instantons in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string compactified on a $K3$.

4.1. A simple example in detail

As is well known, the $SO(32)$ Heterotic theory compactified on a $K3$ requires 24 instantons to cancel the anomaly. Consider the situation when two of the instantons shrink to zero at the same point. As was shown in [24], the effective theory develops a nonperturbative $Sp(2)$ gauge symmetry. In addition, since there are only 22 finite size instantons left, an $SO(10)$ subgroup of the primordial $SO(32)$ (or, more precisely, $\mathrm{Spin}(32)/\mathbb{Z}_2$) that was previously broken to $SO(8)$ is now restored. One can easily calculate that there are 231 neutral hypermultiplets and a single massless tensor multiplet in the six-dimensional spectrum. If we compactify further to four dimensions on a torus and go to the Coulomb phase of the resulting $N = 2$ theory, we obtain rank($Sp(2)$)+rank($SO(10)$)+3=10 massless vector multiplets and 231 massless hypermultiplets. So, if we find the Type IIA dual then the corresponding Calabi–Yau manifold will have $h_{11} = 10$ and $h_{21} = 230$. If we can represent this manifold as a hypersurface in a toric variety, then we should be able to read the $Sp(2) \times SO(10)$ gauge group off the dual polyhedron.

There is indeed a very simple procedure for generating the Calabi–Yau manifold in question torically. As was conjectured in [16], the $SO(32)$ heterotic string on a $K3$ is T-dual to the $E_8 \times E_8$ heterotic string on another $K3$ with instanton numbers $(16, 8)$ which in turn can be described by F-theory compactified on an Calabi–Yau threefold elliptically fibered over $\mathbb{F}_4$. So, what we have to do is to take the Calabi–Yau manifold defined by Table 2.1 for $n = 4$ and introduce an $Sp(2)$ singularity by the methods described in §2. The resulting dual polyhedron then consists of the points displayed in Table 4.1.

Points in the first column of the Table lie in the slice of the polyhedron by the plane $x_1 = 0$, $x_2 = 0$, and this set of points is a copy of the torus of Figure 3.2. Notice also that if we project ∇ onto this plane we obtain the same points. The slice itself forms a two-dimensional reflexive polyhedron describing a torus. We learn from our rules that the threefold is an elliptic fibration which in this case is true by construction. Next, project

$\Pi_B^{-1}(0,0)$	$\Pi_B^{-1}(R_1)$	$\Pi_B^{-1}(R_2)$	$\Pi_B^{-1}(R_3)$	$\Pi_B^{-1}(R_4)$
(0, 0, 2, 3)	(1, 4, 2, 3)	(0, 2, 2, 3)	(-1, 0, 2, 3)	(0, -1, 2, 3)
(0, 0, 1, 2)	(1, 4, 1, 2)	(0, 1, 2, 3)		
(0, 0, 0, 1)	(1, 4, 0, 1)	(0, 2, 1, 2)		
(0, 0, 1, 1)		(0, 1, 1, 2)		
(0, 0, 0, 0)		(0, 1, 1, 1)		
(0, 0, 0, -1)		(0, 1, 0, 1)		
(0, 0, -1, 0)		(0, 1, 0, 0)		

Table 4.1: Points of the dual polyhedron describing the $SO(32)$ heterotic vacuum with two point-like instantons, sorted according to how they project onto the fan of the base of the fibration.

onto the first two coordinates. This is our projection Π_B. As we see from the Table, we obtain six points

$$\Pi_B(\nabla) = \{(0,0),\ (-1,0),\ (0,-1),\ (0,1),\ (0,2),\ (1,4)\},$$

that form the fan of $\mathbb{F}_4$. We conclude that $\mathbb{F}_4$ is the base of the elliptic fibration, which we knew to be true by consruction in this case. Using (3.3) we find that $h_{11} = 2$ for this base, and from (3.4) that there is indeed only one massless tensor multiplet present in the six-dimensional spectrum.

Let us now turn to the enhanced gauge symmetry content of ∇. As was claimed in §2, above each one-dimensional cone/ray of the fan of the base there is a simple factor of the total gauge group. Some of these may be the trivial group, which, for want of better notation, we denote by $SU(1)$. More precisely, the points $\Pi_B^{-1}(R)$, where R is one of the rays of Figure 4.1, give us a simple factor in the enhanced gauge symmetry of the effective theory. In this case there are four such rays.

The points of $\Pi_B^{-1}(R)$, for some ray R, are of the form (ka, kb, c, d) where (c,d) is a point of $\nabla^{\mathcal{E}}$, (a,b) is the first integral point along R and k is a positive integer. The point (c,d) is one of the points of $\nabla^{\mathcal{E}}$ that are denoted pt_i and we associate with (ka, kb, c, d) the point $pt_i^{(k)}$. Consider first $\Pi_B^{-1}(R_2)$. We see from Table 4.1 that

$$\Pi_B^{-1}(R_2) = \{pt_1'', pt_1', pt_2'', pt_2', pt_3', pt_4', pt_5'\} \simeq SO(10),$$

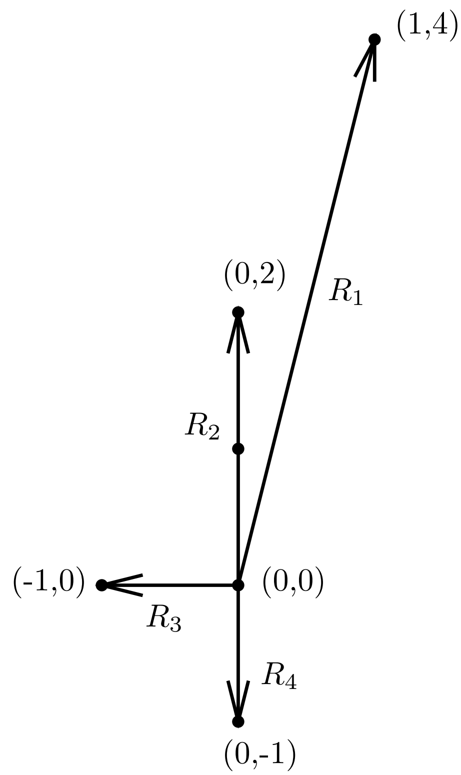

Figure 4.1: The fan of $\mathbb{F}_4$.

the second equivalence following from a comparison with Table 3.1. For the ray R_1 we have

$$\Pi_B^{-1}(R_1) = \{pt'_1, pt'_2, pt'_4\} \simeq Sp(2)$$

while for $\Pi_B^{-1}(R_3)$ and $\Pi_B^{-1}(R_4)$ we find $\{pt'_1\}$ corresponding to the trivial group $SU(1)$. Thus, combining the groups, we find the gauge group $SO(10) \times Sp(2)$, in precise agreement with our expectations.

Now we can study the gauge content of F-Theory compactified on the mirrors of the Calabi–Yau manifolds of Ref. [10]. We are going to make use of the fact that these manifolds are elliptic fibrations, moreover, the polyhedron describing the generic fiber is visible in the dual polyhedron as an injection. This property is shared by the direct polyhedron, hence the mirror Calabi–Yau manifold is also an elliptic fibration. Therefore, we can take the corresponding direct polyhedra regarding them as dual polyhedra of the mirrors, and read off the tensor and vector multiplet spectra. Since the h_{21}'s of the original Calabi–Yau manifold are rather large, so are h_{11}'s of their mirrors, suggesting that the enhanced

gauge symmetry in four dimensions we are about to uncover is bigger than what we are accustomed to. In particular, if heterotic duals of these models exist, most of the gauge symmetry in question is bound to have a nonperturbative origin. We describe a few examples below. The complete results for this class of models are listed in the Appendix.

4.2. The mirror of the manifold with Hodge numbers (3,243)

The Calabi–Yau threefold defined by the data in Table 2.1 for $n = 0$ has $(h_{11}, h_{21}) = (3, 243)$ and provides a dual to the compactification of heterotic strings with instanton numbers $(12, 12)$ in the two E_8's. There is no enhanced gauge symmetry in six dimensions. The mirror Calabi–Yau manifold has $(h_{11}, h_{21}) = (243, 3)$ of course which tells us that $\mathrm{rank}(G) + n_T = 241$. Using our methods we find that Σ_B has 96 one-dimensional cones which seems to suggest that we have 93 massless tensors in the spectrum. Then when we begin to "sort out" the points corresponding to vector multiplets we find that there are 8 E_8 factors but in 4 of them the point which projects on the pt_5 in Fig. 3.2 becomes 'relevant' as opposed to the usual E_8 'top' described in [10] where it was interior to a facet. This is the case which is encountered in almost all in the examples considered in this section. We will interpret the extra 'relevant' points as additional tensor multiplets (see §5). So the six-dimensional theory corresponding to the mirror has gauge group

$$G = E_8^8 \times F_4^8 \times G_2^{16} \times SU(2)^{16} \quad \text{and} \quad n_T = 97.$$

Now unhiggs an $SU(2)$ subgroup of the first E_8 on the heterotic side. The corresponding Calabi–Yau manifold has $(h_{11}, h_{21}) = (4, 214)$ and the six-dimensional theory being a perturbative heterotic vacuum still has just one massless tensor. The h_{11} of the dual Calabi–Yau manifold is now 214 suggesting that $\mathrm{rank}(G) + n_T = 212$. We have 82 one-dimensional cones in Σ_B and two sets of E_8 points with one extra 'relevant' point. That brings the number of tensors to $82 - 3 + 2 = 81$ and in addition there is a gauge group of rank 131: $E_8^5 \times E_7^3 \times F_4^6 \times G_2^{12} \times SO(7)^2 \times SU(2)^{16}$.

4.3. The mirror of the manifold with Hodge numbers (11,491)

Consider the model dual to the heterotic vacuum obtained by compactification with instanton numbers $(24, 0)$. There is an unbroken E_8 and the Hodge numbers are $(h_{11}, h_{21}) = (11, 491)$. This manifold is interesting because it has the maximal h_{21} encountered in those

examples and, moreover, the maximal h_{21} found in [25,26] for hypersurfaces in toric varieties. The mirror thus yields the biggest h_{11} suggesting that this may be the case in which the maximal possible gauge symmetry is found in type II (and, perhaps, heterotic) compactifications to six or four dimensions, and was studied in [14]. On applying the algorithm we find the gauge group of rank 296 mentioned in the introduction

$$G_{296} = E_8^{17} \times F_4^{16} \times G_2^{32} \times SU(2)^{32} \quad \text{and} \quad n_T = 193.$$

4.4. The self-mirror manifold with Hodge numbers (251,251)

The previous example is related in an interesting way to the self-mirror manifold with Hodge numbers $(251, 251)$. This is the largest value of the Hodge numbers among the self-mirror examples listed in [25,26] (see also the interesting comments on this space and its relation to the previous one in [18]). Furthermore, the sum $h_{11} + h_{21}$ is the same as for the previous example. We find a gauge group of rank 152 which is

$$G_{152} = E_8^9 \times F_4^8 \times G_2^{16} \times SU(2)^{16} \text{ and } n_T = 97.$$

Note, as a curiosity, that this gauge group is the 'square root' of the product of the gauge groups in the preceding example, *i.e.*, $G_{152} \times G_{152} = E_8 \times G_{296}$, and the numbers of tensors are also related : $2 \times 97 = 1 + 193$.

The algorithm for finding the gauge content applies equally well to elliptic fourfolds, the only difference from the threefold case being that the base is now a three dimensional toric variety. Since F-theory compactified on a fourfold yeilds a four-dimensional theory, there are no tensor multiplets, and the rank of the gauge group is simply [27,28]

$$\text{rank}(G) = h_{11} - h_{11}(B) - 1 + h_{21}(B) \ .$$

For the examples we study, the base is a toric variety, so that $h_{21}(B) = 0$. Also, $h_{11}(B)$ is related to the number of rays in the fan of the base in a simple way. The last two examples that we give here are the fourfold analogues of the last two examples [18]. The relevant Hodge numbers are now h_{11} and h_{31} and these two fourfolds are distinguished among the Fermat hypersurfaces in weighted projective spaces by having $h_{11} + h_{31}$ as large as possible. In one case $h_{31} - h_{11}$ is also a maximum while in the other the manifold is self-mirror (and hence has $h_{11} = h_{31}$). We include these examples here to show that for fourfolds the groups can become very large indeed. Also, it was pointed out in [29] that there is an additional contribution to the total gauge group coming from three-branes which are necessary for anomaly cancellation. Generically, this additional factor is $U(1)^{|\chi|/24}$, provided there are no instantons in the seven-branes.

4.5. The elliptic fourfold $\mathbb{P}_5^{(1,1,84,516,1204,1806)}[3612]$

This manifold has $h_{11} = 252$ and $h_{31} = 303148$. If we apply the algorithm to Δ we obtain the group G_{152} of the previous example. Applying the algorithm to ∇, we obtain a group of rank 121328

$$G_{121328} \;=\; E_8^{2561} \times F_4^{7576} \times G_2^{20168} \times SU(2)^{30200} \;.$$

4.6. The elliptic fourfold $\mathbb{P}_5^{(1,1806,75894,466206,1087814,1631721)}[3263442]$

This manifold is self mirror with Hodge numbers $h_{11} = h_{31} = 151700$. Both the manifold and its mirror correspond to the same group, G_{60740}, of the rank indicated. Rather than write this group out explicitly, we simply note that this fourfold and the previous one manifest the same curious group property as their threefold analogues

$$G_{60740} \times G_{60740} \;=\; G_{152} \times G_{121328} \;.$$

5. **Subtleties Revisited**

While presenting the algorithm, we mentioned certain subtleties that arise in the toric picture. Here we give prescriptions for handling some of the common difficulties that we encounter. This is best done by giving examples. Consider the $n = 3$ model, which has gauge group $SU(3)$, and Hodge numbers $(5, 251)$. The dual polyhedron for this threefold contains the top that we identify with $SU(2)$. So naively, one might expect the effective theory to be an $SU(2)$ gauge theory. This is wrong, since the correct answer is $SU(3)$. However, the number of non-toric deformations can be readily counted, and is seen to be 1. Thus the "missing" rank of the gauge group appears non-torically. If we were to add a point to the top to make the $SU(3)$ structure explicit, then we find that the Hodge numbers are unchanged, but the number of non-toric deformations is now zero. We interpret this to mean that the same manifold can be described by different polyhedra, but for our purposes, the most useful description is the one with zero non-toric deformations. The cases we discuss below are all ultimately dealt with in the same way. In all the cases that we have analysed in detail, we have been able to reduce the correction to zero by adding points to the polyhedron while holding the Hodge numbers fixed. We have made the assumption that this process does not change the manifold and we have found that in all the cases we have analysed in detail, this enables the algorithm to produce the correct answer.

Another interesting situation occurs when we unhiggs an E_8 gauge group. For concreteness, take $n = 0$. We find Hodge numbers $(23, 143)$. In this case δ is 11. We know that unhiggsing E_8 causes all the instantons in it to become tensor multiplets, so we expect 12 extra tensors. Our first instinct is to say that the extra tensors are encoded in the non-toric data, and that it should be possible to add points to the polyhedron to make $\delta = 0$. This is, in fact, true. We can add 11 points to the dual polyhedron in such a way as to leave the Hodge numbers invariant and reduce the number of non-toric deformations to zero, as well as make 11 extra tensors explicit in the polyhedron. However, we are still short one tensor multiplet. The extra tensor multiplet is manifested as pt'_5 which is interior to a codimension one face of the $K3$ polyhedron but *not* interior to a codimension one face of the threefold polyhedron. Points for which this is true can, by an enumeration of cases, be interpreted as tensor multiplets or group factors. Similar situations result when we unhiggs E_{6b}, E_{7b}, $SO(12)$, $SO(9)\times SU(2)$, $G_2\times SU(3)$ and $SU(6)_c$, when points pt'_7, pt'_6, pt''_6, pt'''_6, pt'_5 and pt'_7 respectively are relevant (*i.e.,* they lie in codimension-2 faces), and

they are also accompanied by non-zero values of the non-toric correction δ. These are the "blue points" of the figures of [10]. In all these cases, they correspond to extra tensors. Curiously, we can add extra points to the polyhedra for each of these groups in such a way that the Hodge numbers do not change, but δ vanishes and the "blue point" becomes irrelevant since it now lies in the interior of a codimension one face of the threefold polyhedron. Other more complicated situations occasionally arise. However, in all cases that we have examined we are able to apply the algorithm unambiguously after adding points appropriately to the polyhedron.

6. **Discussion**

In this paper, we have presented an extension of the dictionary between toric geometry data and and the spectrum of type IIA strings (F-theory) compactified to four (six) dimensions on a Calabi–Yau threefold described as a hypersurface in a toric variety. Specifically, we were able to find the enhanced gauge symmetry as well as the number of massless tensor multiplets (in six dimensions) observed in the resulting vacuum. Apart from the models dual to perturbative heterotic vacua and those resulting from "simple" point-like instantons which was analyzed previously, our methods permit the analysis of cases with a large gauge group as well as a large number of massless tensors. We present many such examples. The algorithm generalises readily to the case of elliptic fourfolds. We are currently studying the gauge content of a large class of such fourfolds, and hope to report on our progress in future. Our method can only be applied fully if the correction term, δ, vanishes. Otherwise there remains a contribution, δ, to be apportioned between the rank of the group and the number of tensor multiplets. In all the cases that we have analysed in detail, it has proved possible to add points to the polyhedron so as to reduce the correction term to zero while holding the Hodge numbers fixed. It seems improbable that this will always be possible. One obvious omission from our toric geometry—physics dictionary is the information about charged matter content. We hope to report on the progress in this direction in future.

Another interesting question to answer, we believe, would be, given such a compactification of the type II theory, what is its heterotic dual. It is fairly obvious that in order to provide such a gauge symmetry by means of a heterotic compactification on a $K3$, it would be necessary to combine singularities of the gauge bundle with singularities of the internal manifold. It would be very interesting to be able to specify the exact singularity structure which yields the duals to our models.

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A. **Appendix: Tables of Gauge Groups**

We list tables of groups that we have identified using the methods discussed above. These groups were calculated for the manifolds of Ref. [10] and their mirrors. The tables list the groups, the rank of the groups, the number of tensor multiplets and the relevant Hodge number. To conserve space, we list the groups in subscripted form, thus, SU_3^2 represents $SU(3)^2$. For the original manifolds, the gauge and matter content is completely known from heterotic/Type II duality, hence it is easy to identify the non-toric deformations as either contributions to the gauge group or the number of tensors. For the mirror manifolds, however, we cannot always identify the non-toric corrections with certainty, since the gauge content of the heterotic dual (if any) is not known. Thus, our knowledge of the gauge content of the theories encoded by the mirror manifolds is incomplete in those cases where $\tilde{\delta}$ is non-zero. We therefore also list the value of $\tilde{\delta}$ for each of the mirrors.

$n = 0$, Groups $SU_1 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\widetilde{\delta}$	h_{21}
SU_1	0	1	3	$E_8^8 F_4^8 G_2^{16} SU_2^{16}$	144	97	0	243
SU_2	1	1	4	$E_7^3 E_8^5 F_4^6 G_2^{12} SO_7^2 SU_2^{16}$	131	81	0	214
SU_{2b}	1	1	4	$E_7^3 E_8^5 F_4^6 G_2^{12} SO_7^2 SU_2^{16}$	131	81	0	214
SU_{2c}	1	2	5	$E_8^5 F_4^4 G_2^{10} SO_5^2 SO_9^2 SO_{11}^2 SO_{13} SU_2^{12}$	116	67	0	185
SU_{2d}	1	3	6	$E_8^5 F_4^5 G_2^{10} SU_2^{11}$	91	63	0	156
SU_3	2	1	5	$E_6^3 E_8^5 F_4^6 G_2^{10} SU_2^{10} SU_3^4$	120	75	0	197
SU_{3b}	2	1	5	$E_6^3 E_8^5 F_4^6 G_2^{10} SU_2^{10} SU_3^4$	120	75	0	197
SU_{3c}	2	3	7	$E_8^5 F_4^4 G_2^9 SU_2^{12} SU_3$	88	61	0	151
SO_5	2	1	5	$E_8^5 F_4^4 G_2^{10} SO_5^2 SO_9^2 SO_{11}^3 SU_2^{12}$	115	67	1	185
G_2	2	1	5	$E_8^5 F_4^9 G_2^{10} SU_2^{14}$	110	75	10	197
SU_2^2	2	1	5	$E_8^5 F_4^4 G_2^{10} SO_5^2 SO_9^2 SO_{11}^2 SO_{12} SU_2^{12}$	116	67	0	185
$SU_2 SU_{2b}$	2	2	6	$E_8^5 F_4^4 G_2^{10} SO_9 SU_2^{11}$	91	63	0	156
SU_4	3	1	6	$E_8^5 F_4^4 G_2^{10} SO_9^2 SO_{10}^3 SU_2^{14}$	113	67	0	182
SO_7	3	1	6	$E_8^5 F_4^4 G_2^{10} SO_9^5 SU_2^{14}$	110	67	3	182
Sp_3	3	1	6	$E_8^5 F_4^4 G_2^{11} SU_2^{11}$	89	63	2	156
$SU_2 SU_3$	3	1	6	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3^2 SU_4^2 SU_5^2 SU_6$	105	61	0	168
$SU_3 SU_{2b}$	3	1	6	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3^2 SU_4^2 SU_5^2 SU_6$	105	61	0	168
$SU_3 SU_{2c}$	3	1	6	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3^2 SU_4^2 SU_5^2 SU_6$	105	61	0	168
$SO_5 SU_2$	3	1	6	$E_8^5 F_4^4 G_2^{10} SO_7 SU_2^{11}$	90	63	1	156
$G_2 SU_2$	3	1	6	$E_8^5 F_4^4 G_2^8 SO_5^4 Sp_3 SU_2^{12}$	95	61	10	168
SU_5	4	1	7	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3^2 SU_4^2 SU_5^3$	104	61	0	167
SO_9	4	1	7	$E_8^5 F_4^4 G_2^{10} SO_7^5 SU_2^{10}$	101	67	5	175
F_4	4	1	7	$E_8^5 F_4^4 G_2^{15} SU_2^{10}$	96	67	10	175
SU_3^2	4	1	7	$E_8^5 F_4^4 G_2^8 SU_2^{12} SU_3^2$	88	61	0	151
$G_2 SU_3$	4	1	7	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3$	84	55	10	151
$SU_2 SU_4$	4	1	7	$E_8^5 F_4^4 G_2^8 SU_2^{11} SU_3^2 SU_4$	90	61	0	153
$SO_7 SU_2$	4	1	7	$E_8^5 F_4^4 G_2^8 SO_5 SU_2^{13}$	87	61	3	153
SU_6	5	1	8	$E_8^5 F_4^4 G_2^8 SU_2^{11} SU_3^3$	89	61	0	152
SU_{6b}	5	1	8	$E_8^5 F_4^4 G_2^8 SU_2^{13} SU_3$	87	61	0	150
SU_{6c}	5	3	10	$E_8^5 F_4^4 G_2^8 SU_2^{11} SU_3$	85	61	0	148
SO_{10}	5	1	8	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3^2 SU_4^5$	101	61	0	164
SO_{11}	5	1	8	$E_8^5 F_4^4 G_2^8 SO_5^5 SU_2^{12}$	94	61	7	164
$SO_9 SU_2$	5	1	8	$E_8^5 F_4^4 G_2^8 SU_2^{10}$	82	57	5	146
$F_4 SU_2$	5	1	8	$E_8^5 F_4^4 G_2^8 SU_2^9$	81	53	10	146
SO_{12}	6	1	9	$E_8^5 F_4^4 G_2^8 SU_2^{14}$	86	61	0	149
SO_{13}	6	5	13	$E_8^5 F_4^4 G_2^8 SU_2^9$	81	57	5	145
E_6	6	1	9	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3^7$	96	61	0	159
E_{6b}	6	7	15	$E_8^5 F_4^4 G_2^8 SU_2^{10} SU_3$	84	61	0	147
E_7	7	1	10	$E_8^5 F_4^4 G_2^8 SU_2^{17}$	89	61	0	152
E_{7b}	7	9	18	$E_8^5 F_4^4 G_2^8 SU_2^9$	81	61	0	144
E_8	8	13	23	$E_8^5 F_4^4 G_2^8 SU_2^8$	80	61	0	143

$n = 1$, Groups $SU_1 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	0	1	3	$E_8^8 F_4^8 G_2^{16} SU_2^{16}$	144	97	0	243
SU_2	1	1	4	$E_7^4 E_8^4 F_4^5 G_2^{10} SO_7^3 SU_2^{16}$	125	75	0	202
SU_{2b}	1	1	4	$E_7^4 E_8^4 F_4^5 G_2^{10} SO_7^3 SU_2^{16}$	125	75	0	202
SU_{2c}	1	1	4	$E_8^4 F_4^3 G_2^8 SO_5^2 SO_9^2 SO_{11}^2 SO_{13}^2 Sp_3 SU_2^{10}$	107	57	0	166
SU_{2d}	1	1	4	$E_8^4 F_4^5 G_2^8 SU_2^{10}$	78	55	1	136
SU_3	2	1	5	$E_6^4 E_8^4 F_4^5 G_2^8 SU_2^8 SU_3^5$	110	67	0	179
SU_{3b}	2	1	5	$E_6^4 E_8^4 F_4^5 G_2^8 SU_2^8 SU_3^5$	110	67	0	179
SU_{3c}	2	1	5	$E_8^4 F_4^3 G_2^8 SU_2^{10} SU_3$	72	51	0	125
SO_5	2	1	5	$E_8^4 F_4^3 G_2^8 SO_5^3 SO_9^2 SO_{11}^4 SU_2^{10}$	104	57	2	165
G_2	2	1	5	$E_8^4 F_4^9 G_2^8 SU_2^{13}$	97	67	13	179
SU_2^2	2	1	5	$E_8^4 F_4^3 G_2^8 SO_5^3 SO_9^2 SO_{11}^2 SO_{12}^2 SU_2^{10}$	106	57	0	165
$SU_2 SU_{2b}$	2	1	5	$E_8^4 F_4^3 G_2^8 SO_9^2 SU_2^{10}$	78	53	0	133
SU_4	3	1	6	$E_8^4 F_4^3 G_2^8 SO_9^2 SO_{10}^4 SU_2^{13}$	101	57	0	160
SO_7	3	1	6	$E_8^4 F_4^3 G_2^8 SO_9^6 SU_2^{13}$	97	57	4	160
Sp_3	3	1	6	$E_8^4 F_4^3 G_2^{10} SU_2^9$	73	53	4	132
$SU_2 SU_3$	3	1	6	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3^2 SU_4^2 SU_5^2 SU_6^2$	92	50	0	144
$SU_3 SU_{2b}$	3	1	6	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3^2 SU_4^2 SU_5^2 SU_6^2$	92	50	0	144
$SU_3 SU_{2c}$	3	1	6	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3^2 SU_4^2 SU_5^2 SU_6^2$	92	50	0	144
$SO_5 SU_2$	3	1	6	$E_8^4 F_4^3 G_2^8 SO_7^2 SU_2^9$	75	53	2	132
$G_2 SU_2$	3	1	6	$E_8^4 F_4^3 G_2^6 SO_5^4 Sp_3^2 SU_2^{10}$	80	50	12	144
SU_5	4	1	7	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3^2 SU_4^2 SU_5^4$	90	50	0	142
SO_9	4	1	7	$E_8^4 F_4^3 G_2^8 SO_7^6 SU_2^8$	86	57	6	151
F_4	4	1	7	$E_8^4 F_4^3 G_2^{14} SU_2^8$	80	57	12	151
SU_3^2	4	1	7	$E_8^4 F_4^3 G_2^6 SU_2^{10} SU_3^3$	72	50	0	124
$G_2 SU_3$	4	2	8	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3$	66	44	10	122
$SU_2 SU_4$	4	1	7	$E_8^4 F_4^3 G_2^6 SU_2^9 SU_3^2 SU_4^2$	75	50	0	127
$SO_7 SU_2$	4	1	7	$E_8^4 F_4^3 G_2^6 SO_5^2 SU_2^{11}$	71	50	4	127
SU_6	5	1	8	$E_8^4 F_4^3 G_2^6 SU_2^9 SU_3^4$	73	50	0	125
SU_{6b}	5	1	8	$E_8^4 F_4^3 G_2^6 SU_2^{12} SU_3$	70	50	0	122
SU_{6c}	5	4	11	$E_8^4 F_4^3 G_2^6 SU_2^9 SU_3$	67	50	0	119
SO_{10}	5	1	8	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3^2 SU_4^6$	86	50	0	138
SO_{11}	5	1	8	$E_8^4 F_4^3 G_2^6 SO_5^6 SU_2^{10}$	78	50	8	138
$SO_9 SU_2$	5	1	8	$E_8^4 F_4^3 G_2^6 SU_2^9$	65	46	5	118
$F_4 SU_2$	5	2	9	$E_8^4 F_4^3 G_2^6 SU_2^7$	63	42	10	117
SO_{12}	6	1	9	$E_8^4 F_4^3 G_2^6 SU_2^{13}$	69	50	0	121
SO_{13}	6	6	14	$E_8^4 F_4^3 G_2^6 SU_2^7$	63	46	5	116
E_6	6	1	9	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3^8$	80	50	0	132
E_{6b}	6	8	16	$E_8^4 F_4^3 G_2^6 SU_2^8 SU_3$	66	50	0	118
E_7	7	1	10	$E_8^4 F_4^3 G_2^6 SU_2^{16}$	72	50	0	124
E_{7b}	7	10	19	$E_8^4 F_4^3 G_2^6 SU_2^7$	63	50	0	115
E_8	8	14	24	$E_8^4 F_4^3 G_2^6 SU_2^6$	62	50	0	114

$n = 2$, Groups $SU_1 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	0	1	3	$E_8^8 F_4^8 G_2^{16} SU_2^{16}$	144	96	1	243
SU_2	1	1	4	$E_7^5 E_8^3 F_4^4 G_2^8 SO_7^4 SU_2^{16}$	119	68	1	190
SU_{2b}	1	1	4	$E_7^5 E_8^3 F_4^4 G_2^8 SO_7^4 SU_2^{16}$	119	68	1	190
SU_{2c}	1	1	4	$E_8^3 F_4^2 G_2^6 SO_5^2 SO_9^2 SO_{11}^2 SO_{13}^3 Sp_3^2 SU_2^8$	98	46	2	148
SU_{2d}	1	1	4	$E_8^3 F_4^5 G_2^6 SU_2^9$	65	46	5	118
SU_3	2	1	5	$E_6^5 E_8^3 F_4^4 G_2^6 SU_2^6 SU_3^6$	100	58	1	161
SU_{3b}	2	1	5	$E_6^5 E_8^3 F_4^4 G_2^6 SU_2^6 SU_3^6$	100	58	1	161
SU_{3c}	2	1	5	$E_8^3 F_4^2 G_2^7 SU_2^8 SU_3$	56	40	3	101
SO_5	2	1	5	$E_8^3 F_4^2 G_2^6 SO_5^4 SO_9^2 SO_{11}^5 SU_2^8$	93	46	4	145
G_2	2	1	5	$E_8^3 F_4^9 G_2^6 SU_2^{12}$	84	58	17	161
SU_2^2	2	1	5	$E_8^3 F_4^2 G_2^6 SO_5^4 SO_9^2 SO_{11}^2 SO_{12}^3 SU_2^8$	96	46	1	145
$SU_2 SU_{2b}$	2	1	5	$E_8^3 F_4^2 G_2^6 SO_9^3 SU_2^9$	65	42	2	111
SU_4	3	1	6	$E_8^3 F_4^2 G_2^6 SO_9^2 SO_{10}^5 SU_2^{12}$	89	46	1	138
SO_7	3	1	6	$E_8^3 F_4^2 G_2^6 SO_9^7 SU_2^{12}$	84	46	6	138
Sp_3	3	1	6	$E_8^3 F_4^2 G_2^9 SU_2^7$	57	42	7	108
$SU_2 SU_3$	3	1	6	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3^2 SU_4^2 SU_5^2 SU_6^3$	79	38	1	120
$SU_3 SU_{2b}$	3	1	6	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3^2 SU_4^2 SU_5^2 SU_6^3$	79	38	1	120
$SU_3 SU_{2c}$	3	1	6	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3^2 SU_4^2 SU_5^2 SU_6^3$	79	38	1	120
$SO_5 SU_2$	3	1	6	$E_8^3 F_4^2 G_2^6 SO_7^3 SU_2^7$	60	42	4	108
$G_2 SU_2$	3	1	6	$E_8^3 F_4^2 G_2^4 SO_5^4 Sp_3^3 SU_2^8$	65	38	15	120
SU_5	4	1	7	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3^2 SU_4^2 SU_5^5$	76	38	1	117
SO_9	4	1	7	$E_8^3 F_4^2 G_2^6 SO_7^7 SU_2^6$	71	46	8	127
F_4	4	1	7	$E_8^3 F_4^2 G_2^{13} SU_2^6$	64	46	15	127
SU_3^2	4	1	7	$E_8^3 F_4^2 G_2^4 SU_2^8 SU_3^4$	56	38	1	97
$G_2 SU_3$	4	3	9	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3$	48	32	11	93
$SU_2 SU_4$	4	1	7	$E_8^3 F_4^2 G_2^4 SU_2^7 SU_3^2 SU_4^3$	60	38	1	101
$SO_7 SU_2$	4	1	7	$E_8^3 F_4^2 G_2^4 SO_5^3 SU_2^9$	55	38	6	101
SU_6	5	1	8	$E_8^3 F_4^2 G_2^4 SU_2^7 SU_3^5$	57	38	1	98
SU_{6b}	5	1	8	$E_8^3 F_4^2 G_2^4 SU_2^{11} SU_3$	53	38	1	94
SU_{6c}	5	5	12	$E_8^3 F_4^2 G_2^4 SU_2^7 SU_3$	49	38	1	90
SO_{10}	5	1	8	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3^2 SU_4^7$	71	38	1	112
SO_{11}	5	1	8	$E_8^3 F_4^2 G_2^4 SO_5^7 SU_2^8$	62	38	10	112
$SO_9 SU_2$	5	1	8	$E_8^3 F_4^2 G_2^4 SU_2^8$	48	34	6	90
$F_4 SU_2$	5	3	10	$E_8^3 F_4^2 G_2^4 SU_2^5$	45	30	11	88
SO_{12}	6	1	9	$E_8^3 F_4^2 G_2^4 SU_2^{12}$	52	38	1	93
SO_{13}	6	7	15	$E_8^3 F_4^2 G_2^4 SU_2^5$	45	34	6	87
E_6	6	1	9	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3^9$	64	38	1	105
E_{6b}	6	9	17	$E_8^3 F_4^2 G_2^4 SU_2^6 SU_3$	48	38	1	89
E_7	7	1	10	$E_8^3 F_4^2 G_2^4 SU_2^{15}$	55	38	1	96
E_{7b}	7	11	20	$E_8^3 F_4^2 G_2^4 SU_2^5$	45	38	1	86
E_8	8	15	25	$E_8^3 F_4^2 G_2^4 SU_2^4$	44	38	1	85

$n\ =\ 3$, Groups $SU_3 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	2	1	5	$E_8^8F_4^9G_2^{17}SU_2^{16}$	150	99	0	251
SU_2	3	1	6	$E_7^6E_8^2F_4^4G_2^7SO_7^5SU_2^{16}$	119	65	0	186
SU_{2b}	3	1	6	$E_7^6E_8^2F_4^4G_2^7SO_7^5SU_2^{16}$	119	65	0	186
SU_{2c}	3	1	6	$E_8^2F_4^2G_2^5SO_5^2SO_9^2SO_{11}^2SO_{13}^4Sp_3^3SU_2^6$	95	39	2	138
SU_{2d}	3	1	6	$E_8^2F_4^6G_2^5SU_2^8$	58	41	7	108
SU_3	4	1	7	$E_6^6E_8^2F_4^4G_2^5SU_2^4SU_3^7$	96	53	0	151
SU_{3b}	4	1	7	$E_6^6E_8^2F_4^4G_2^5SU_2^4SU_3^7$	96	53	0	151
SU_{3c}	4	1	7	$E_8^2F_4^2G_2^7SU_2^6SU_3$	46	33	4	85
SO_5	4	1	7	$E_8^2F_4^2G_2^5SO_5^5SO_9^2SO_{11}^6SU_2^6$	88	39	4	133
G_2	4	1	7	$E_8^2F_4^{10}G_2^5SU_2^{11}$	77	53	19	151
SU_2^2	4	1	7	$E_8^2F_4^2G_2^5SO_5^5SO_9^2SO_{11}^2SO_{12}^4SU_2^6$	92	39	0	133
SU_2SU_{2b}	4	1	7	$E_8^2F_4^2G_2^5SO_9^4SU_2^8$	58	35	2	97
SU_4	5	1	8	$E_8^2F_4^2G_2^5SO_9^2SO_{10}^6SU_2^{11}$	83	39	0	124
SO_7	5	1	8	$E_8^2F_4^2G_2^5SO_9^8SU_2^{11}$	77	39	6	124
Sp_3	5	1	8	$E_8^2F_4^2G_2^9SU_2^5$	47	35	8	92
SU_2SU_3	5	1	8	$E_8^2F_4^2G_2^3SU_2^4SU_3^2SU_4^2SU_5^2SU_6^4$	72	30	0	104
SU_3SU_{2b}	5	1	8	$E_8^2F_4^2G_2^3SU_2^4SU_3^2SU_4^2SU_5^2SU_6^4$	72	30	0	104
SU_3SU_{2c}	5	1	8	$E_8^2F_4^2G_2^3SU_2^4SU_3^2SU_4^2SU_5^2SU_6^4$	72	30	0	104
SO_5SU_2	5	1	8	$E_8^2F_4^2G_2^5SO_7^4SU_2^5$	51	35	4	92
G_2SU_2	5	1	8	$E_8^2F_4^2G_2^3SO_5^4Sp_3^4SU_2^6$	56	30	16	104
SU_5	6	1	9	$E_8^2F_4^2G_2^3SU_2^4SU_3^2SU_4^2SU_5^6$	68	30	0	100
SO_9	6	1	9	$E_8^2F_4^2G_2^5SO_7^8SU_2^4$	62	39	8	111
F_4	6	1	9	$E_8^2F_4^2G_2^{13}SU_2^4$	54	39	16	111
SU_3^2	6	1	9	$E_8^2F_4^2G_2^3SU_2^6SU_3^5$	46	30	0	78
G_2SU_3	6	4	12	$E_8^2F_4^2G_2^3SU_2^4SU_3$	36	24	10	72
SU_2SU_4	6	1	9	$E_8^2F_4^2G_2^3SU_2^5SU_3^2SU_4^4$	51	30	0	83
SO_7SU_2	6	1	9	$E_8^2F_4^2G_2^3SO_5^4SU_2^7$	45	30	6	83
SU_6	7	1	10	$E_8^2F_4^2G_2^3SU_2^5SU_3^6$	47	30	0	79
SU_{6b}	7	1	10	$E_8^2F_4^2G_2^3SU_2^{10}SU_3$	42	30	0	74
SU_{6c}	7	6	15	$E_8^2F_4^2G_2^3SU_2^5SU_3$	37	30	0	69
SO_{10}	7	1	10	$E_8^2F_4^2G_2^3SU_2^4SU_3^2SU_4^8$	62	30	0	94
SO_{11}	7	1	10	$E_8^2F_4^2G_2^3SO_5^8SU_2^6$	52	30	10	94
SO_9SU_2	7	1	10	$E_8^2F_4^2G_2^3SU_2^7$	37	26	5	70
F_4SU_2	7	4	13	$E_8^2F_4^2G_2^3SU_2^3$	33	22	10	67
SO_{12}	8	1	11	$E_8^2F_4^2G_2^3SU_2^{11}$	41	30	0	73
SO_{13}	8	8	18	$E_8^2F_4^2G_2^3SU_2^3$	33	26	5	66
E_6	8	1	11	$E_8^2F_4^2G_2^3SU_2^4SU_3^{10}$	54	30	0	86
E_{6b}	8	10	20	$E_8^2F_4^2G_2^3SU_2^4SU_3$	36	30	0	68
E_7	9	1	12	$E_8^2F_4^2G_2^3SU_2^{14}$	44	30	0	76
E_{7b}	9	12	23	$E_8^2F_4^2G_2^3SU_2^3$	33	30	0	65
E_8	10	16	28	$E_8^2F_4^2G_2^3SU_2^2$	32	30	0	64

$n = 4$, Groups $SO_8 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\mathrm{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	4	1	7	$E_8^9 F_4^9 G_2^{18} SU_2^{18}$	162	107	0	271
SU_2	5	1	8	$E_7^7 E_8^2 F_4^3 G_2^6 SO_7^6 SU_2^{18}$	125	67	0	194
SU_{2b}	5	1	8	$E_7^7 E_8^2 F_4^3 G_2^6 SO_7^6 SU_2^{18}$	125	67	0	194
SU_{2c}	5	1	8	$E_8^2 F_4 G_2^4 SO_5^2 SO_9^2 SO_{11}^2 SO_{13}^5 Sp_3^4 SU_2^6$	98	37	3	140
SU_{2d}	5	1	8	$E_8^2 F_4^6 G_2^4 SU_2^9$	57	41	10	110
SU_3	6	1	9	$E_6^7 E_8^2 F_4^3 G_2^4 SU_2^4 SU_3^8$	98	53	0	153
SU_{3b}	6	1	9	$E_6^7 E_8^2 F_4^3 G_2^4 SU_2^4 SU_3^8$	98	53	0	153
SU_{3c}	6	1	9	$E_8^2 F_4 G_2^7 SU_2^6 SU_3$	42	31	6	81
SO_5	6	1	9	$E_8^2 F_4 G_2^4 SO_5^6 SO_9^2 SO_{11}^7 SU_2^6$	89	37	5	133
G_2	6	1	9	$E_8^2 F_4^{10} G_2^4 SU_2^{12}$	76	53	22	153
SU_2^2	6	1	9	$E_8^2 F_4 G_2^4 SO_5^6 SO_9^2 SO_{11}^2 SO_{12}^5 SU_2^6$	94	37	0	133
$SU_2 SU_{2b}$	6	1	9	$E_8^2 F_4 G_2^4 SO_9^5 SU_2^9$	57	33	3	95
SU_4	7	1	10	$E_8^2 F_4 G_2^4 SO_9^2 SO_{10}^7 SU_2^{12}$	83	37	0	122
SO_7	7	1	10	$E_8^2 F_4 G_2^4 SO_9^9 SU_2^{12}$	76	37	7	122
Sp_3	7	1	10	$E_8^2 F_4 G_2^9 SU_2^5$	43	33	10	88
$SU_2 SU_3$	7	1	10	$E_8^2 F_4 G_2^2 SU_2^4 SU_3^2 SU_4^2 SU_5^2 SU_6^5$	71	27	0	100
$SU_3 SU_{2b}$	7	1	10	$E_8^2 F_4 G_2^2 SU_2^4 SU_3^2 SU_4^2 SU_5^2 SU_6^5$	71	27	0	100
$SU_3 SU_{2c}$	7	1	10	$E_8^2 F_4 G_2^2 SU_2^4 SU_3^2 SU_4^2 SU_5^2 SU_6^5$	71	27	0	100
$SO_5 SU_2$	7	1	10	$E_8^2 F_4 G_2^4 SO_7^5 SU_2^5$	48	33	5	88
$G_2 SU_2$	7	1	10	$E_8^2 F_4 G_2^2 SO_5^4 Sp_3^5 SU_2^6$	53	27	18	100
SU_5	8	1	11	$E_8^2 F_4 G_2^2 SU_2^4 SU_3^2 SU_4^2 SU_5^7$	66	27	0	95
SO_9	8	1	11	$E_8^2 F_4 G_2^4 SO_7^9 SU_2^4$	59	37	9	107
F_4	8	1	11	$E_8^2 F_4 G_2^{13} SU_2^4$	50	37	18	107
SU_3^2	8	1	11	$E_8^2 F_4 G_2^2 SU_2^6 SU_3^6$	42	27	0	71
$G_2 SU_3$	8	5	15	$E_8^2 F_4 G_2^2 SU_2^4 SU_3$	30	21	10	63
$SU_2 SU_4$	8	1	11	$E_8^2 F_4 G_2^2 SU_2^5 SU_3^2 SU_4^5$	48	27	0	77
$SO_7 SU_2$	8	1	11	$E_8^2 F_4 G_2^2 SO_5^5 SU_2^7$	41	27	7	77
SU_6	9	1	12	$E_8^2 F_4 G_2^2 SU_2^5 SU_3^7$	43	27	0	72
SU_{6b}	9	1	12	$E_8^2 F_4 G_2^2 SU_2^{11} SU_3$	37	27	0	66
SU_{6c}	9	7	18	$E_8^2 F_4 G_2^2 SU_2^5 SU_3$	31	27	0	60
SO_{10}	9	1	12	$E_8^2 F_4 G_2^2 SU_2^4 SU_3^2 SU_4^9$	59	27	0	88
SO_{11}	9	1	12	$E_8^2 F_4 G_2^2 SO_5^9 SU_2^6$	48	27	11	88
$SO_9 SU_2$	9	1	12	$E_8^2 F_4 G_2^2 SU_2^8$	32	23	5	62
$F_4 SU_2$	9	5	16	$E_8^2 F_4 G_2^2 SU_2^3$	27	19	10	58
SO_{12}	10	1	13	$E_8^2 F_4 G_2^2 SU_2^{12}$	36	27	0	65
SO_{13}	10	9	21	$E_8^2 F_4 G_2^2 SU_2^3$	27	23	5	57
E_6	10	1	13	$E_8^2 F_4 G_2^2 SU_2^4 SU_3^{11}$	50	27	0	79
E_{6b}	10	11	23	$E_8^2 F_4 G_2^2 SU_2^4 SU_3$	30	27	0	59
E_7	11	1	14	$E_8^2 F_4 G_2^2 SU_2^{15}$	39	27	0	68
E_{7b}	11	13	26	$E_8^2 F_4 G_2^2 SU_2^3$	27	27	0	56
E_8	12	17	31	$E_8^2 F_4 G_2^2 SU_2^2$	26	27	0	55

$n \;=\; 5,\quad$ Groups $F_4 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	4	1	7	$E_8^{10}F_4^9G_2^{20}SU_2^{20}$	176	117	0	295
SU_2	5	1	8	$E_7^8E_8^2F_4^2G_2^6SO_7^7SU_2^{20}$	133	71	0	206
SU_{2b}	5	1	8	$E_7^8E_8^2F_4^2G_2^6SO_7^7SU_2^{20}$	133	71	0	206
SU_{2c}	5	1	8	$E_8^2G_2^4SO_5^2SO_9^2SO_{11}^2SO_{13}^6Sp_3^5SU_2^6$	103	37	4	146
SU_{2d}	5	1	8	$E_8^2F_4^6G_2^4SU_2^{10}$	58	43	13	116
SU_3	6	1	9	$E_6^8E_8^2F_4^2G_2^4SU_2^4SU_3^9$	102	55	0	159
SU_{3b}	6	1	9	$E_6^8E_8^2F_4^2G_2^4SU_2^4SU_3^9$	102	55	0	159
SU_{3c}	6	1	9	$E_8^2G_2^8SU_2^6SU_3$	40	31	8	81
SO_5	6	1	9	$E_8^2G_2^4SO_5^7SO_9^2SO_{11}^8SU_2^6$	92	37	6	137
G_2	6	1	9	$E_8^2F_4^{10}G_2^4SU_2^{13}$	77	55	25	159
SU_2^2	6	1	9	$E_8^2G_2^4SO_5^7SO_9^2SO_{11}^2SO_{12}^6SU_2^6$	98	37	0	137
SU_2SU_{2b}	6	1	9	$E_8^2G_2^4SO_9^6SU_2^{10}$	58	33	4	97
SU_4	7	1	10	$E_8^2G_2^4SO_9^2SO_{10}^8SU_2^{13}$	85	37	0	124
SO_7	7	1	10	$E_8^2G_2^4SO_9^{10}SU_2^{13}$	77	37	8	124
Sp_3	7	1	10	$E_8^2G_2^{10}SU_2^5$	41	33	12	88
SU_2SU_3	7	1	10	$E_8^2G_2^2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^6$	72	26	0	100
SU_3SU_{2b}	7	1	10	$E_8^2G_2^2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^6$	72	26	0	100
SU_3SU_{2c}	7	1	10	$E_8^2G_2^2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^6$	72	26	0	100
SO_5SU_2	7	1	10	$E_8^2G_2^4SO_7^6SU_2^5$	47	33	6	88
G_2SU_2	7	1	10	$E_8^2G_2^2SO_5^4Sp_3^6SU_2^6$	52	26	20	100
SU_5	8	1	11	$E_8^2G_2^2SU_2^4SU_3^2SU_4^2SU_5^8$	66	26	0	94
SO_9	8	1	11	$E_8^2G_2^4SO_7^{10}SU_2^4$	58	37	10	107
F_4	8	1	11	$E_8^2G_2^{14}SU_2^4$	48	37	20	107
SU_3^2	8	1	11	$E_8^2G_2^2SU_2^6SU_3^7$	40	26	0	68
G_2SU_3	8	6	16	$E_8^2G_2^2SU_2^4SU_3$	26	20	10	58
SU_2SU_4	8	1	11	$E_8^2G_2^2SU_2^5SU_3^2SU_4^6$	47	26	0	75
SO_7SU_2	8	1	11	$E_8^2G_2^2SO_5^6SU_2^7$	39	26	8	75
SU_6	9	1	12	$E_8^2G_2^2SU_2^5SU_3^8$	41	26	0	69
SU_{6b}	9	1	12	$E_8^2G_2^2SU_2^{12}SU_3$	34	26	0	62
SU_{6c}	9	8	19	$E_8^2G_2^2SU_2^5SU_3$	27	26	0	55
SO_{10}	9	1	12	$E_8^2G_2^2SU_2^4SU_3^2SU_4^{10}$	58	26	0	86
SO_{11}	9	1	12	$E_8^2G_2^2SO_5^{10}SU_2^6$	46	26	12	86
SO_9SU_2	9	1	12	$E_8^2G_2^2SU_2^9$	29	22	5	58
F_4SU_2	9	6	17	$E_8^2G_2^2SU_2^3$	23	18	10	53
SO_{12}	10	1	13	$E_8^2G_2^2SU_2^{13}$	33	26	0	61
SO_{13}	10	10	22	$E_8^2G_2^2SU_2^3$	23	22	5	52
E_6	10	1	13	$E_8^2G_2^2SU_2^4SU_3^{12}$	48	26	0	76
E_{6b}	10	12	24	$E_8^2G_2^2SU_2^4SU_3$	26	26	0	54
E_7	11	1	14	$E_8^2G_2^2SU_2^{16}$	36	26	0	64
E_{7b}	11	14	27	$E_8^2G_2^2SU_2^3$	23	26	0	51
E_8	12	18	32	$E_8^2G_2^2SU_2^2$	22	26	0	50

$n\ =\ 6$, Groups $E_6 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	6	1	9	$E_8^{11}F_4^{10}G_2^{21}SU_2^{22}$	192	127	0	321
SU_2	7	1	10	$E_7^9E_8^2F_4^2G_2^5SO_7^8SU_2^{22}$	143	75	0	220
SU_{2b}	7	1	10	$E_7^9E_8^2F_4^2G_2^5SO_7^8SU_2^{22}$	143	75	0	220
SU_{2c}	7	1	10	$E_8^2G_2^3SO_5^2SO_9^2SO_{11}^2SO_{13}^7Sp_3^6SU_2^6$	110	37	5	154
SU_{2d}	7	1	10	$E_8^2F_4^7G_2^3SU_2^{11}$	61	45	16	124
SU_3	8	1	11	$E_6^9E_8^2F_4^2G_2^3SU_2^4SU_3^{10}$	108	57	0	167
SU_{3b}	8	1	11	$E_6^9E_8^2F_4^2G_2^3SU_2^4SU_3^{10}$	108	57	0	167
SU_{3c}	8	1	11	$E_8^2G_2^8SU_2^6SU_3$	40	31	10	83
SO_5	8	1	11	$E_8^2G_2^3SO_5^8SO_9^2SO_{11}^9SU_2^6$	97	37	7	143
G_2	8	1	11	$E_8^2F_4^{11}G_2^3SU_2^{14}$	80	57	28	167
SU_2^2	8	1	11	$E_8^2G_2^3SO_5^8SO_9^2SO_{11}^2SO_{12}^7SU_2^6$	104	37	0	143
SU_2SU_{2b}	8	1	11	$E_8^2G_2^3SO_9^7SU_2^{11}$	61	33	5	101
SU_4	9	1	12	$E_8^2G_2^3SO_9^2SO_{10}^9SU_2^{14}$	89	37	0	128
SO_7	9	1	12	$E_8^2G_2^3SO_9^{11}SU_2^{14}$	80	37	9	128
Sp_3	9	1	12	$E_8^2G_2^{10}SU_2^5$	41	33	14	90
SU_2SU_3	9	1	12	$E_8^2G_2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^7$	75	25	0	102
SU_3SU_{2b}	9	1	12	$E_8^2G_2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^7$	75	25	0	102
SU_3SU_{2c}	9	1	12	$E_8^2G_2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^7$	75	25	0	102
SO_5SU_2	9	1	12	$E_8^2G_2^3SO_7^7SU_2^5$	48	33	7	90
G_2SU_2	9	1	12	$E_8^2G_2SO_5^4Sp_3^7SU_2^6$	53	25	22	102
SU_5	10	1	13	$E_8^2G_2SU_2^4SU_3^2SU_4^2SU_5^9$	68	25	0	95
SO_9	10	1	13	$E_8^2G_2^3SO_7^{11}SU_2^4$	59	37	11	109
F_4	10	1	13	$E_8^2G_2^{14}SU_2^4$	48	37	22	109
SU_3^2	10	1	13	$E_8^2G_2SU_2^6SU_3^8$	40	25	0	67
G_2SU_3	10	7	19	$E_8^2G_2SU_2^4SU_3$	24	19	10	55
SU_2SU_4	10	1	13	$E_8^2G_2SU_2^5SU_3^2SU_4^7$	48	25	0	75
SO_7SU_2	10	1	13	$E_8^2G_2SO_5^7SU_2^7$	39	25	9	75
SU_6	11	1	14	$E_8^2G_2SU_2^5SU_3^9$	41	25	0	68
SU_{6b}	11	1	14	$E_8^2G_2SU_2^{13}SU_3$	33	25	0	60
SU_{6c}	11	9	22	$E_8^2G_2SU_2^5SU_3$	25	25	0	52
SO_{10}	11	1	14	$E_8^2G_2SU_2^4SU_3^2SU_4^{11}$	59	25	0	86
SO_{11}	11	1	14	$E_8^2G_2SO_5^{11}SU_2^6$	46	25	13	86
SO_9SU_2	11	1	14	$E_8^2G_2SU_2^{10}$	28	21	5	56
F_4SU_2	11	7	20	$E_8^2G_2SU_2^3$	21	17	10	50
SO_{12}	12	1	15	$E_8^2G_2SU_2^{14}$	32	25	0	59
SO_{13}	12	11	25	$E_8^2G_2SU_2^3$	21	21	5	49
E_6	12	1	15	$E_8^2G_2SU_2^4SU_3^{13}$	48	25	0	75
E_{6b}	12	13	27	$E_8^2G_2SU_2^4SU_3$	24	25	0	51
E_7	13	1	16	$E_8^2G_2SU_2^{17}$	35	25	0	62
E_{7b}	13	15	30	$E_8^2G_2SU_2^3$	21	25	0	48
E_8	14	19	35	$E_8^2G_2SU_2^2$	20	25	0	47

$n = 7$, Groups $E_7 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\mathrm{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	7	1	10	$E_8^{12}F_4^{11}G_2^{22}SU_2^{24}$	208	138	0	348
SU_2	8	1	11	$E_7^{10}E_8^2F_4^2G_2^4SO_7^9SU_2^{24}$	153	80	0	235
SU_{2b}	8	1	11	$E_7^{10}E_8^2F_4^2G_2^4SO_7^9SU_2^{24}$	153	80	0	235
SU_{2c}	8	1	11	$E_8^2G_2^2SO_5^2SO_9^2SO_{11}^2SO_{13}^8Sp_3^7SU_2^6$	117	38	6	163
SU_{2d}	8	1	11	$E_8^2F_4^8G_2^2SU_2^{12}$	64	48	19	133
SU_3	9	1	12	$E_6^{10}E_8^2F_4^2G_2^2SU_2^4SU_3^{11}$	114	60	0	176
SU_{3b}	9	1	12	$E_6^{10}E_8^2F_4^2G_2^2SU_2^4SU_3^{11}$	114	60	0	176
SU_{3c}	9	1	12	$E_8^2G_2^8SU_2^6SU_3$	40	32	12	86
SO_5	9	1	12	$E_8^2G_2^2SO_5^9SO_9^2SO_{11}^{10}SU_2^6$	102	38	8	150
G_2	9	1	12	$E_8^2F_4^{12}G_2^2SU_2^{15}$	83	60	31	176
SU_2^2	9	1	12	$E_8^2G_2^2SO_5^9SO_9^2SO_{11}^2SO_{12}^8SU_2^6$	110	38	0	150
SU_2SU_{2b}	9	1	12	$E_8^2G_2^2SO_9^8SU_2^{12}$	64	34	6	106
SU_4	10	1	13	$E_8^2G_2^2SO_9^2SO_{10}^{10}SU_2^{15}$	93	38	0	133
SO_7	10	1	13	$E_8^2G_2^2SO_9^{12}SU_2^{15}$	83	38	10	133
Sp_3	10	1	13	$E_8^2G_2^{10}SU_2^5$	41	34	16	93
SU_2SU_3	10	1	13	$E_8^2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^8$	78	25	0	105
SU_3SU_{2b}	10	1	13	$E_8^2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^8$	78	25	0	105
SU_3SU_{2c}	10	1	13	$E_8^2SU_2^4SU_3^2SU_4^2SU_5^2SU_6^8$	78	25	0	105
SO_5SU_2	10	1	13	$E_8^2G_2^2SO_7^8SU_2^5$	49	34	8	93
G_2SU_2	10	1	13	$E_8^2SO_5^4Sp_3^8SU_2^6$	54	25	24	105
SU_5	11	1	14	$E_8^2SU_2^4SU_3^2SU_4^2SU_5^{10}$	70	25	0	97
SO_9	11	1	14	$E_8^2G_2^2SO_7^{12}SU_2^4$	60	38	12	112
F_4	11	1	14	$E_8^2G_2^{14}SU_2^4$	48	38	24	112
SU_3^2	11	1	14	$E_8^2SU_2^6SU_3^9$	40	25	0	67
G_2SU_3	11	8	21	$E_8^2SU_2^4SU_3$	22	19	10	53
SU_2SU_4	11	1	14	$E_8^2SU_2^5SU_3^2SU_4^8$	49	25	0	76
SO_7SU_2	11	1	14	$E_8^2SO_5^8SU_2^7$	39	25	10	76
SU_6	12	1	15	$E_8^2SU_2^5SU_3^{10}$	41	25	0	68
SU_{6b}	12	1	15	$E_8^2SU_2^{14}SU_3$	32	25	0	59
SU_{6c}	12	10	24	$E_8^2SU_2^5SU_3$	23	25	0	50
SO_{10}	12	1	15	$E_8^2SU_2^4SU_3^2SU_4^{12}$	60	25	0	87
SO_{11}	12	1	15	$E_8^2SO_5^{12}SU_2^6$	46	25	14	87
SO_9SU_2	12	1	15	$E_8^2SU_2^{11}$	27	21	5	55
F_4SU_2	12	8	22	$E_8^2SU_2^3$	19	17	10	48
SO_{12}	13	1	16	$E_8^2SU_2^{15}$	31	25	0	58
SO_{13}	13	12	27	$E_8^2SU_2^3$	19	21	5	47
E_6	13	1	16	$E_8^2SU_2^4SU_3^{14}$	48	25	0	75
E_{6b}	13	14	29	$E_8^2SU_2^4SU_3$	22	25	0	49
E_7	14	1	17	$E_8^2SU_2^{18}$	34	25	0	61
E_{7b}	14	16	32	$E_8^2SU_2^3$	19	25	0	46
E_8	15	20	37	$E_8^2SU_2^2$	18	25	0	45

$n\ =\ 8$, Groups $E_7 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	7	1	10	$E_8^{13}F_4^{12}G_2^{24}SU_2^{25}$	225	149	0	376
SU_2	8	1	11	$E_7^{11}E_8^2F_4^2G_2^4SO_7^{10}SU_2^{25}$	164	85	0	251
SU_{2b}	8	1	11	$E_7^{11}E_8^2F_4^2G_2^4SO_7^{10}SU_2^{25}$	164	85	0	251
SU_{2c}	8	1	11	$E_8^2G_2^2SO_5^2SO_9^2SO_{11}^2SO_{13}^9Sp_3^8SU_2^5$	125	39	7	173
SU_{2d}	8	1	11	$E_8^2F_4^9G_2^2SU_2^{12}$	68	51	22	143
SU_3	9	1	12	$E_6^{11}E_8^2F_4^2G_2^2SU_2^3SU_3^{12}$	121	63	0	186
SU_{3b}	9	1	12	$E_6^{11}E_8^2F_4^2G_2^2SU_2^3SU_3^{12}$	121	63	0	186
SU_{3c}	9	1	12	$E_8^2G_2^9SU_2^5SU_3$	41	33	14	90
SO_5	9	1	12	$E_8^2G_2^2SO_5^{10}SO_9^2SO_{11}^{11}SU_2^5$	108	39	9	158
G_2	9	1	12	$E_8^2F_4^{13}G_2^2SU_2^{15}$	87	63	34	186
SU_2^2	9	1	12	$E_8^2G_2^2SO_5^{10}SO_9^2SO_{11}^2SO_{12}^9SU_2^5$	117	39	0	158
SU_2SU_{2b}	9	1	12	$E_8^2G_2^2SO_9^9SU_2^{12}$	68	35	7	112
SU_4	10	1	13	$E_8^2G_2^2SO_9^2SO_{10}^{11}SU_2^{15}$	98	39	0	139
SO_7	10	1	13	$E_8^2G_2^2SO_9^{13}SU_2^{15}$	87	39	11	139
Sp_3	10	1	13	$E_8^2G_2^{11}SU_2^4$	42	35	18	97
SU_2SU_3	10	1	13	$E_8^2SU_2^3SU_3^2SU_4^2SU_5^2SU_6^9$	82	25	0	109
SU_3SU_{2b}	10	1	13	$E_8^2SU_2^3SU_3^2SU_4^2SU_5^2SU_6^9$	82	25	0	109
SU_3SU_{2c}	10	1	13	$E_8^2SU_2^3SU_3^2SU_4^2SU_5^2SU_6^9$	82	25	0	109
SO_5SU_2	10	1	13	$E_8^2G_2^2SO_7^9SU_2^4$	51	35	9	97
G_2SU_2	10	1	13	$E_8^2SO_5^4Sp_3^9SU_2^5$	56	25	26	109
SU_5	11	1	14	$E_8^2SU_2^3SU_3^2SU_4^2SU_5^{11}$	73	25	0	100
SO_9	11	1	14	$E_8^2G_2^2SO_7^{13}SU_2^3$	62	39	13	116
F_4	11	1	14	$E_8^2G_2^{15}SU_2^3$	49	39	26	116
SU_3^2	11	1	14	$E_8^2SU_2^5SU_3^{10}$	41	25	0	68
G_2SU_3	11	9	22	$E_8^2SU_2^3SU_3$	21	19	10	52
SU_2SU_4	11	1	14	$E_8^2SU_2^4SU_3^2SU_4^9$	51	25	0	78
SO_7SU_2	11	1	14	$E_8^2SO_5^9SU_2^6$	40	25	11	78
SU_6	12	1	15	$E_8^2SU_2^4SU_3^{11}$	42	25	0	69
SU_{6b}	12	1	15	$E_8^2SU_2^{14}SU_3$	32	25	0	59
SU_{6c}	12	11	25	$E_8^2SU_2^4SU_3$	22	25	0	49
SO_{10}	12	1	15	$E_8^2SU_2^3SU_3^2SU_4^{13}$	62	25	0	89
SO_{11}	12	1	15	$E_8^2SO_5^{13}SU_2^5$	47	25	15	89
SO_9SU_2	12	1	15	$E_8^2SU_2^{11}$	27	21	5	55
F_4SU_2	12	9	23	$E_8^2SU_2^2$	18	17	10	47
SO_{12}	13	1	16	$E_8^2SU_2^{15}$	31	25	0	58
SO_{13}	13	13	28	$E_8^2SU_2^2$	18	21	5	46
E_6	13	1	16	$E_8^2SU_2^3SU_3^{15}$	49	25	0	76
E_{6b}	13	15	30	$E_8^2SU_2^3SU_3$	21	25	0	48
E_7	14	1	17	$E_8^2SU_2^{18}$	34	25	0	61
E_{7b}	14	17	33	$E_8^2SU_2^2$	18	25	0	45
E_8	15	21	38	$E_8^2SU_2$	17	25	0	44

$n = 9$, Groups $E_8 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\mathrm{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	8	4	14	$E_8^{14}F_4^{13}G_2^{26}SU_2^{26}$	242	160	0	404
SU_2	9	4	15	$E_7^{12}E_8^2F_4^2G_2^4SO_7^{11}SU_2^{26}$	175	90	0	267
SU_{2b}	9	4	15	$E_7^{12}E_8^2F_4^2G_2^4SO_7^{11}SU_2^{26}$	175	90	0	267
SU_{2c}	9	4	15	$E_8^2G_2^2SO_5^2SO_9^2SO_{11}^2SO_{13}^{10}Sp_3^9SU_2^4$	133	40	8	183
SU_{2d}	9	4	15	$E_8^2F_4^{10}G_2^2SU_2^{12}$	72	54	25	153
SU_3	10	4	16	$E_6^{12}E_8^2F_4^2G_2^2SU_2^2SU_3^{13}$	128	66	0	196
SU_{3b}	10	4	16	$E_6^{12}E_8^2F_4^2G_2^2SU_2^2SU_3^{13}$	128	66	0	196
SU_{3c}	10	4	16	$E_8^2G_2^{10}SU_2^4SU_3$	42	34	16	94
SO_5	10	4	16	$E_8^2G_2^2SO_5^{11}SO_9^2SO_{11}^{12}SU_2^4$	114	40	10	166
G_2	10	4	16	$E_8^2F_4^{14}G_2^2SU_2^{15}$	91	66	37	196
SU_2^2	10	4	16	$E_8^2G_2^2SO_5^{11}SO_9^2SO_{11}^2SO_{12}^{10}SU_2^4$	124	40	0	166
SU_2SU_{2b}	10	4	16	$E_8^2G_2^2SO_9^{10}SU_2^{12}$	72	36	8	118
SU_4	11	4	17	$E_8^2G_2^2SO_9^2SO_{10}^{12}SU_2^{15}$	103	40	0	145
SO_7	11	4	17	$E_8^2G_2^2SO_9^{14}SU_2^{15}$	91	40	12	145
Sp_3	11	4	17	$E_8^2G_2^{12}SU_2^3$	43	36	20	101
SU_2SU_3	11	4	17	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{10}$	86	25	0	113
SU_3SU_{2b}	11	4	17	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{10}$	86	25	0	113
SU_3SU_{2c}	11	4	17	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{10}$	86	25	0	113
SO_5SU_2	11	4	17	$E_8^2G_2^2SO_7^{10}SU_2^3$	53	36	10	101
G_2SU_2	11	4	17	$E_8^2SO_5^4Sp_3^{10}SU_2^4$	58	25	28	113
SU_5	12	4	18	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^{12}$	76	25	0	103
SO_9	12	4	18	$E_8^2G_2^2SO_7^{14}SU_2^2$	64	40	14	120
F_4	12	4	18	$E_8^2G_2^{16}SU_2^2$	50	40	28	120
SU_3^2	12	4	18	$E_8^2SU_2^4SU_3^{11}$	42	25	0	69
G_2SU_3	12	13	27	$E_8^2SU_2^2SU_3$	20	19	10	51
SU_2SU_4	12	4	18	$E_8^2SU_2^3SU_3^2SU_4^{10}$	53	25	0	80
SO_7SU_2	12	4	18	$E_8^2SO_5^{10}SU_2^5$	41	25	12	80
SU_6	13	4	19	$E_8^2SU_2^3SU_3^{12}$	43	25	0	70
SU_{6b}	13	4	19	$E_8^2SU_2^{14}SU_3$	32	25	0	59
SU_{6c}	13	15	30	$E_8^2SU_2^3SU_3$	21	25	0	48
SO_{10}	13	4	19	$E_8^2SU_2^2SU_3^2SU_4^{14}$	64	25	0	91
SO_{11}	13	4	19	$E_8^2SO_5^{14}SU_2^4$	48	25	16	91
SO_9SU_2	13	4	19	$E_8^2SU_2^{11}$	27	21	5	55
F_4SU_2	13	13	28	$E_8^2SU_2$	17	17	10	46
SO_{12}	14	4	20	$E_8^2SU_2^{15}$	31	25	0	58
SO_{13}	14	17	33	$E_8^2SU_2$	17	21	5	45
E_6	14	4	20	$E_8^2SU_2^2SU_3^{16}$	50	25	0	77
E_{6b}	14	19	35	$E_8^2SU_2^2SU_3$	20	25	0	47
E_7	15	4	21	$E_8^2SU_2^{18}$	34	25	0	61
E_{7b}	15	21	38	$E_8^2SU_2$	17	25	0	44
E_8	16	25	43	E_8^2	16	25	0	43

$n\ =\ 10$, Groups $E_8 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	8	3	13	$E_8^{15}F_4^{14}G_2^{28}SU_2^{28}$	260	171	0	433
SU_2	9	3	14	$E_7^{13}E_8^2F_4^2G_2^4SO_7^{12}SU_2^{28}$	187	95	0	284
SU_{2b}	9	3	14	$E_7^{13}E_8^2F_4^2G_2^4SO_7^{12}SU_2^{28}$	187	95	0	284
SU_{2c}	9	3	14	$E_8^2G_2^2SO_5^2SO_9^2SO_{11}^2SO_{13}^{11}Sp_3^{10}SU_2^4$	142	41	9	194
SU_{2d}	9	3	14	$E_8^2F_4^{11}G_2^2SU_2^{13}$	77	57	28	164
SU_3	10	3	15	$E_6^{13}E_8^2F_4^2G_2^2SU_2^2SU_3^{14}$	136	69	0	207
SU_{3b}	10	3	15	$E_6^{13}E_8^2F_4^2G_2^2SU_2^2SU_3^{14}$	136	69	0	207
SU_{3c}	10	3	15	$E_8^2G_2^{11}SU_2^4SU_3$	44	35	18	99
SO_5	10	3	15	$E_8^2G_2^2SO_5^{12}SO_9^2SO_{11}^{13}SU_2^4$	121	41	11	175
G_2	10	3	15	$E_8^2F_4^{15}G_2^2SU_2^{16}$	96	69	40	207
SU_2^2	10	3	15	$E_8^2G_2^2SO_5^{12}SO_9^2SO_{11}^2SO_{12}^{11}SU_2^4$	132	41	0	175
SU_2SU_{2b}	10	3	15	$E_8^2G_2^2SO_9^{11}SU_2^{13}$	77	37	9	125
SU_4	11	3	16	$E_8^2G_2^2SO_9^2SO_{10}^{13}SU_2^{16}$	109	41	0	152
SO_7	11	3	16	$E_8^2G_2^2SO_9^{15}SU_2^{16}$	96	41	13	152
Sp_3	11	3	16	$E_8^2G_2^{13}SU_2^3$	45	37	22	106
SU_2SU_3	11	3	16	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{11}$	91	25	0	118
SU_3SU_{2b}	11	3	16	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{11}$	91	25	0	118
SU_3SU_{2c}	11	3	16	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{11}$	91	25	0	118
SO_5SU_2	11	3	16	$E_8^2G_2^2SO_7^{11}SU_2^3$	56	37	11	106
G_2SU_2	11	3	16	$E_8^2SO_5^4Sp_3^{11}SU_2^4$	61	25	30	118
SU_5	12	3	17	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^{13}$	80	25	0	107
SO_9	12	3	17	$E_8^2G_2^2SO_7^{15}SU_2^2$	67	41	15	125
F_4	12	3	17	$E_8^2G_2^{17}SU_2^2$	52	41	30	125
SU_3^2	12	3	17	$E_8^2SU_2^4SU_3^{12}$	44	25	0	71
G_2SU_3	12	13	27	$E_8^2SU_2^2SU_3$	20	19	10	51
SU_2SU_4	12	3	17	$E_8^2SU_2^3SU_3^2SU_4^{11}$	56	25	0	83
SO_7SU_2	12	3	17	$E_8^2SO_5^{11}SU_2^5$	43	25	13	83
SU_6	13	3	18	$E_8^2SU_2^3SU_3^{13}$	45	25	0	72
SU_{6b}	13	3	18	$E_8^2SU_2^{15}SU_3$	33	25	0	60
SU_{6c}	13	15	30	$E_8^2SU_2^3SU_3$	21	25	0	48
SO_{10}	13	3	18	$E_8^2SU_2^2SU_3^2SU_4^{15}$	67	25	0	94
SO_{11}	13	3	18	$E_8^2SO_5^{15}SU_2^4$	50	25	17	94
SO_9SU_2	13	3	18	$E_8^2SU_2^{12}$	28	21	5	56
F_4SU_2	13	13	28	$E_8^2SU_2$	17	17	10	46
SO_{12}	14	3	19	$E_8^2SU_2^{16}$	32	25	0	59
SO_{13}	14	17	33	$E_8^2SU_2$	17	21	5	45
E_6	14	3	19	$E_8^2SU_2^2SU_3^{17}$	52	25	0	79
E_{6b}	14	19	35	$E_8^2SU_2^2SU_3$	20	25	0	47
E_7	15	3	20	$E_8^2SU_2^{19}$	35	25	0	62
E_{7b}	15	21	38	$E_8^2SU_2$	17	25	0	44
E_8	16	25	43	E_8^2	16	25	0	43

$n = 11$, Groups $E_8 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	8	2	12	$E_8^{16} F_4^{15} G_2^{30} SU_2^{30}$	278	182	0	462
SU_2	9	2	13	$E_7^{14} E_8^2 F_4^2 G_2^4 SO_7^{13} SU_2^{30}$	199	100	0	301
SU_{2b}	9	2	13	$E_7^{14} E_8^2 F_4^2 G_2^4 SO_7^{13} SU_2^{30}$	199	100	0	301
SU_{2c}	9	2	13	$E_8^2 G_2^2 SO_5^2 SO_9^2 SO_{11}^2 SO_{13}^{12} Sp_3^{11} SU_2^4$	151	42	10	205
SU_{2d}	9	2	13	$E_8^2 F_4^{12} G_2^2 SU_2^{14}$	82	60	31	175
SU_3	10	2	14	$E_6^{14} E_8^2 F_4^2 G_2^2 SU_2^2 SU_3^{15}$	144	72	0	218
SU_{3b}	10	2	14	$E_6^{14} E_8^2 F_4^2 G_2^2 SU_2^2 SU_3^{15}$	144	72	0	218
SU_{3c}	10	2	14	$E_8^2 G_2^{12} SU_2^4 SU_3$	46	36	20	104
SO_5	10	2	14	$E_8^2 G_2^2 SO_5^{13} SO_9^2 SO_{11}^{14} SU_2^4$	128	42	12	184
G_2	10	2	14	$E_8^2 F_4^{16} G_2^2 SU_2^{17}$	101	72	43	218
SU_2^2	10	2	14	$E_8^2 G_2^2 SO_5^{13} SO_9^2 SO_{11}^2 SO_{12}^{12} SU_2^4$	140	42	0	184
$SU_2 SU_{2b}$	10	2	14	$E_8^2 G_2^2 SO_9^{12} SU_2^{14}$	82	38	10	132
SU_4	11	2	15	$E_8^2 G_2^2 SO_9^2 SO_{10}^{14} SU_2^{17}$	115	42	0	159
SO_7	11	2	15	$E_8^2 G_2^2 SO_9^{16} SU_2^{17}$	101	42	14	159
Sp_3	11	2	15	$E_8^2 G_2^{14} SU_2^3$	47	38	24	111
$SU_2 SU_3$	11	2	15	$E_8^2 SU_2^2 SU_3^2 SU_4^2 SU_5^2 SU_6^{12}$	96	25	0	123
$SU_3 SU_{2b}$	11	2	15	$E_8^2 SU_2^2 SU_3^2 SU_4^2 SU_5^2 SU_6^{12}$	96	25	0	123
$SU_3 SU_{2c}$	11	2	15	$E_8^2 SU_2^2 SU_3^2 SU_4^2 SU_5^2 SU_6^{12}$	96	25	0	123
$SO_5 SU_2$	11	2	15	$E_8^2 G_2^2 SO_7^{12} SU_2^3$	59	38	12	111
$G_2 SU_2$	11	2	15	$E_8^2 SO_5^4 Sp_3^{12} SU_2^4$	64	25	32	123
SU_5	12	2	16	$E_8^2 SU_2^2 SU_3^2 SU_4^2 SU_5^{14}$	84	25	0	111
SO_9	12	2	16	$E_8^2 G_2^2 SO_7^{16} SU_2^2$	70	42	16	130
F_4	12	2	16	$E_8^2 G_2^{18} SU_2^2$	54	42	32	130
SU_3^2	12	2	16	$E_8^2 SU_2^4 SU_3^{13}$	46	25	0	73
$G_2 SU_3$	12	13	27	$E_8^2 SU_2^2 SU_3$	20	19	10	51
$SU_2 SU_4$	12	2	16	$E_8^2 SU_2^3 SU_3^2 SU_4^{12}$	59	25	0	86
$SO_7 SU_2$	12	2	16	$E_8^2 SO_5^{12} SU_2^5$	45	25	14	86
SU_6	13	2	17	$E_8^2 SU_2^3 SU_3^{14}$	47	25	0	74
SU_{6b}	13	2	17	$E_8^2 SU_2^{16} SU_3$	34	25	0	61
SU_{6c}	13	15	30	$E_8^2 SU_2^3 SU_3$	21	25	0	48
SO_{10}	13	2	17	$E_8^2 SU_2^2 SU_3^2 SU_4^{16}$	70	25	0	97
SO_{11}	13	2	17	$E_8^2 SO_5^{16} SU_2^4$	52	25	18	97
$SO_9 SU_2$	13	2	17	$E_8^2 SU_2^{13}$	29	21	5	57
$F_4 SU_2$	13	13	28	$E_8^2 SU_2$	17	17	10	46
SO_{12}	14	2	18	$E_8^2 SU_2^{17}$	33	25	0	60
SO_{13}	14	17	33	$E_8^2 SU_2$	17	21	5	45
E_6	14	2	18	$E_8^2 SU_2^2 SU_3^{18}$	54	25	0	81
E_{6b}	14	19	35	$E_8^2 SU_2^2 SU_3$	20	25	0	47
E_7	15	2	19	$E_8^2 SU_2^{20}$	36	25	0	63
E_{7b}	15	21	38	$E_8^2 SU_2$	17	25	0	44
E_8	16	25	43	E_8^2	16	25	0	43

$n\ =\ 12$, Groups $E_8 \times H$, Mirror Groups $\widetilde{H}$								
H	rk	n_T	h_{11}	$\widetilde{H}$	$\widetilde{\text{rk}}$	$\tilde{n}_T$	$\tilde{\delta}$	h_{21}
SU_1	8	1	11	$E_8^{17}F_4^{16}G_2^{32}SU_2^{32}$	296	193	0	491
SU_2	9	1	12	$E_7^{15}E_8^2F_4^2G_2^4SO_7^{14}SU_2^{32}$	211	105	0	318
SU_{2b}	9	1	12	$E_7^{15}E_8^2F_4^2G_2^4SO_7^{14}SU_2^{32}$	211	105	0	318
SU_{2c}	9	1	12	$E_8^2G_2^2SO_5^2SO_9^2SO_{11}^2SO_{13}^{13}Sp_3^{12}SU_2^4$	160	43	11	216
SU_{2d}	9	1	12	$E_8^2F_4^{13}G_2^2SU_2^{15}$	87	63	34	186
SU_3	10	1	13	$E_6^{15}E_8^2F_4^2G_2^2SU_2^2SU_3^{16}$	152	75	0	229
SU_{3b}	10	1	13	$E_6^{15}E_8^2F_4^2G_2^2SU_2^2SU_3^{16}$	152	75	0	229
SU_{3c}	10	1	13	$E_8^2G_2^{13}SU_2^4SU_3$	48	37	22	109
SO_5	10	1	13	$E_8^2G_2^2SO_5^{14}SO_9^2SO_{11}^{15}SU_2^4$	135	43	13	193
G_2	10	1	13	$E_8^2F_4^{17}G_2^2SU_2^{18}$	106	75	46	229
SU_2^2	10	1	13	$E_8^2G_2^2SO_5^{14}SO_9^2SO_{11}^2SO_{12}^{13}SU_2^4$	148	43	0	193
SU_2SU_{2b}	10	1	13	$E_8^2G_2^2SO_9^{13}SU_2^{15}$	87	39	11	139
SU_4	11	1	14	$E_8^2G_2^2SO_9^2SO_{10}^{15}SU_2^{18}$	121	43	0	166
SO_7	11	1	14	$E_8^2G_2^2SO_9^{17}SU_2^{18}$	106	43	15	166
Sp_3	11	1	14	$E_8^2G_2^{15}SU_2^3$	49	39	26	116
SU_2SU_3	11	1	14	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{13}$	101	25	0	128
SU_3SU_{2b}	11	1	14	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{13}$	101	25	0	128
SU_3SU_{2c}	11	1	14	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^2SU_6^{13}$	101	25	0	128
SO_5SU_2	11	1	14	$E_8^2G_2^2SO_7^{13}SU_2^3$	62	39	13	116
G_2SU_2	11	1	14	$E_8^2SO_5^4Sp_3^{13}SU_2^4$	67	25	34	128
SU_5	12	1	15	$E_8^2SU_2^2SU_3^2SU_4^2SU_5^{15}$	88	25	0	115
SO_9	12	1	15	$E_8^2G_2^2SO_7^{17}SU_2^2$	73	43	17	135
F_4	12	1	15	$E_8^2G_2^{19}SU_2^2$	56	43	34	135
SU_3^2	12	1	15	$E_8^2SU_2^4SU_3^{14}$	48	25	0	75
G_2SU_3	12	13	27	$E_8^2SU_2^2SU_3$	20	19	10	51
SU_2SU_4	12	1	15	$E_8^2SU_2^3SU_3^2SU_4^{13}$	62	25	0	89
SO_7SU_2	12	1	15	$E_8^2SO_5^{13}SU_2^5$	47	25	15	89
SU_6	13	1	16	$E_8^2SU_2^3SU_3^{15}$	49	25	0	76
SU_{6b}	13	1	16	$E_8^2SU_2^{17}SU_3$	35	25	0	62
SU_{6c}	13	15	30	$E_8^2SU_2^3SU_3$	21	25	0	48
SO_{10}	13	1	16	$E_8^2SU_2^2SU_3^2SU_4^{17}$	73	25	0	100
SO_{11}	13	1	16	$E_8^2SO_5^{17}SU_2^4$	54	25	19	100
SO_9SU_2	13	1	16	$E_8^2SU_2^{14}$	30	21	5	58
F_4SU_2	13	13	28	$E_8^2SU_2$	17	17	10	46
SO_{12}	14	1	17	$E_8^2SU_2^{18}$	34	25	0	61
SO_{13}	14	17	33	$E_8^2SU_2$	17	21	5	45
E_6	14	1	17	$E_8^2SU_2^2SU_3^{19}$	56	25	0	83
E_{6b}	14	19	35	$E_8^2SU_2^2SU_3$	20	25	0	47
E_7	15	1	18	$E_8^2SU_2^{21}$	37	25	0	64
E_{7b}	15	21	38	$E_8^2SU_2$	17	25	0	44
E_8	16	25	43	E_8^2	16	25	0	43

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